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CANCELLATION OF LOW-RANK VECTOR BUNDLES

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The purpose of this paper is to show that even though vector bundles cannot in general be cancelled from direct sums (Whitney sums), in certain low-rank situations vector bundles can be cancelled at the expense of complexifying or quaternionifying the remaining terms. To be specific, let λ, ξ_1, ξ_2 be vector bundles over a paracompact space X, such that $\lambda \oplus \xi_1 \cong \lambda \oplus \xi_2$. First assume that these are real vector bundles. If λ is a line bundle of finite type, then the complexifications of ξ_1 and ξ_2 are isomorphic, and hence $2\xi_1 \cong 2\xi_2$ (where $2\xi_i$ denotes the direct sum of two copies of ξ_i), while if λ is a direct sum of two line bundles of finite type, then the quaternionifications of ξ_1 and ξ_2 are isomorphic, and hence $4\xi_1 \cong 4\xi_2$. Now assume that these are complex vector bundles. If λ is the complexification of a real line bundle of finite type (in particular, λ could be a trivial complex vector bundle of rank 1), then the quaternionifications of ξ_1 and ξ_2 are isomorphic, and hence $\xi_1 \oplus \bar{\xi}_1 \cong \xi_2 \oplus \bar{\xi}_2$ (where $\bar{\xi}_i$ denotes the conjugate vector bundle to ξ_i). These results are independent of the dimension of the space X, and also independent of the dimensions of the fibres of ξ_1 and ξ_2. The same results also hold for smooth vector bundles over a smooth manifold.

I. Background on vector bundles. Let $\mathbf{F}$ be one of $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$. We work with $\mathbf{F}$-vector bundles over a topological space X in the same generality as Swan [9], so that we do not require the dimensions of the fibres of a vector bundle ξ to be constant. (However, the local triviality condition on ξ forces the fibre-dimension of ξ to be locally constant, and hence the fibre-dimension will be constant if X is connected.) A vector bundle ξ over X is said to be of *finite type* provided that there exists a finite open covering $U_1, \ldots, U_n$ of X such that the restriction of ξ to each U_i is trivial. (In particular, this places a bound on the fibre-dimension of ξ.) Of course if X is compact, then all vector bundles over X are of finite type.

The basic mechanism for algebraic investigations of $\mathbf{F}$-vector bundles over X is the relationship between such vector bundles and modules over the ring $C(X, \mathbf{F})$ of continuous $\mathbf{F}$-valued functions on X. This relationship is effected through the section functor Γ, which assigns to each $\mathbf{F}$-vector bundle ξ the $C(X, \mathbf{F})$-module $\Gamma(\xi)$ of continuous sections of ξ, and which assigns to each $\mathbf{F}$-vector bundle map $f: \xi \to \eta$ the induced $C(X, \mathbf{F})$-module homomorphism $\Gamma(f): \Gamma(\xi) \to \Gamma(\eta)$. (In the case where $\mathbf{F} = \mathbf{H}$, a choice of right versus left vector spaces and modules is required for consistency. As we prefer to write homomorphisms on the left, let us stipulate that all our