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ENUMERATION OF THE QUASISIMPLICIAL 3-SPHERES AND 4-POLYTOPES WITH EIGHT VERTICES

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A complete enumeration is given for quasisimplicial 3-spheres with eight vertices. It is found that there are 661 spheres, and it is proved that precisely 20 of these are not polytopal.

1. Introduction. This work is the first part of an enumeration of all the combinatorial 3-spheres and the 4-polytopes with 8 vertices. It is devoted to the quasisimplicial cases; that is, to those 3-spheres and 4-polytopes all of whose facets are simplicial.

In the last two decades extensive work has been done on the enumeration of simplicial 3-spheres and simplicial 4-polytopes with 8 vertices ([**15**], [**10**], [**1**]), with 9 vertices ([**4**], [**6**], [**7**], [**8**]) and with 10 vertices (the neighborly cases only, see [**3**]). With regard to the general (that is, not necessarily simplicial) cases, Kleinschmidt [**16**] proved that every d-sphere with up to $d + 4$ vertices is polytopal, Reif ([**19**]) prepared a detailed list of the 31 4-polytopes with 7 vertices and a partial list (containing some 1050 cases) of 3-spheres with 8 vertices, and Perles ([**14**, pages 114, 424]) determined the number of d-polytopes with $d + 3$ vertices for $d \leq 6$.

The simplicial 3-spheres with 8 vertices are well known ([**15**], [**10**], [**1**]). There are 39, exactly two of which are non-polytopal. The pyramidal 3-spheres with 8 vertices are also well known, based on the determination by Hermes, quoted in [**13**] and [**14**, page 424], of the 3-polytopes with 7 vertices. Obviously they all are polytopal, and their number is 34. Five of them are quasisimplicial.

In the present work we find all the non-simplicial, non-pyramidal quasisimplicial 3-spheres with 8 vertices. It turns out that there are 617, exactly 18 of which are not polytopal. Therefore the total number of quasisimplicial 3-spheres with 8 vertices is 661 of which 20 are not polytopal. Our 18 non-polytopal spheres are described in detail in Table 1. A detailed description of all the 617 cases is of course beyond the scope of a single paper. The full catalogue of these cases can be obtained upon request from the second author.