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SEPARATION, RETRACTIONS AND HOMOTOPY EXTENSION IN SEMIALGEBRAIC SPACES

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We consider an affine semialgebraic space M over a real closed field R. Tietze's extension theorem holds in M. Every closed semialgebraic subset A of M is a strong deformation retract of a semialgebraic neighbourhood Z in M, and (M, A) has the homotopy extension property. If A is locally complete then Z can be chosen as a mapping cylinder.

Let R be a real closed field. In [$\mathbf{DK}_2$] we developed a theory of semialgebraic spaces and mappings over R, and in [**D**], [$\mathbf{DK}_3$], we showed that for an affine semialgebraic space M over R there exist reasonable homology and cohomology groups $H_r(M, G)$, $H^r(M, G)$ with coefficients in an arbitrary abelian group G which coincide with the classical singular homology and cohomology groups in the case $R = \mathbf{R}$. There should also exist a reasonable homotopy theory for such spaces, and this theory — as well as homology theory — should certainly be useful in algebraic geometry over R and over $R(\sqrt{-1})$.

The present paper serves as ground work for semialgebraic homotopy theory. As in topological homotopy theory, we have to make sure, for example, that under sufficiently general assumptions a subspace A of M has a semialgebraic neighbourhood U in M which retracts to A, and that the pair (M, A) has the homotopy extension property for semialgebraic maps. It turns out that in the category of affine semialgebraic spaces these matters are even nicer than in topology.

In §1 we show that affine semialgebraic spaces have separation properties similar to paracompact spaces, a fact which is particularly useful for the sheaf theory of these spaces (a topic we do not consider here, cf. [**D**]). §2 is devoted to a proof of our central result, Theorem 2.1 below, which in a slightly weaker and simpler form says the following.

THEOREM 1. *Any closed semialgebraic subset A of an affine semialgebraic space M has an open semialgebraic neighbourhood U in M such that A is a strong deformation retract of both U and of the closure $\overline{U}$ of U in M (in the semialgebraic sense).*

We feel that this result is a remarkable instance of the good-natured behaviour of the affine semialgebraic category over an arbitrary real closed field R.