PACIFIC JOURNAL OF MATHEMATICS
Vol 114, No 1, 1984

SOLUTIONS OF CERTAIN QUATERNARY QUADRATIC SYSTEMS

DUNCAN A. BUELL AND RICHARD H. HUDSON

For primes $p = qf + 1$, Diophantine systems of the type

$$(1)\qquad 16p^k = x^2 + 2qu^2 + 2qv^2 + qw^2, \quad (x, u, v, w, p) = 1,$$
$$xw = av^2 - 2buv - au^2,$$

have been studied by Dickson, Whiteman, Lehmer, Hasse, Zee, and Muskat and Zee. Virtually all these studies have centered on the special cases $q = 5$, 13 (the correspondence between the system (1) when $q = 5$ and the well-known system introduced by Dickson is discussed in §3). For $q = 13$, 29, 37, 53, and 61, Hudson and Williams have proved that (1) has exactly eight solutions when $k = 1$. For values of $q \equiv 5 \pmod 8$ $= a^2 + b^2$ for which the class number of the imaginary cyclic quartic field $K = Q(i\sqrt{2q + 2a\sqrt{q}})$ is greater than one, (1) may or may not be solvable when $k = 1$. In §5 we examine families of values of q and p for which there are eight solutions of (1) when $k = 1$ independent of any class number considerations. The existence of such families is somewhat surprising, as is the fact that the question of solvability for these families is independent of the primality of p or q (clearly we must have $q = a^2 + b^2$) or the restriction $p = qf + 1$. Indeed the entire study of systems of type (1) is restricted in the literature to primes $p = qf + 1$ artificially, as any completely general study should treat all primes $p = qf + r$, $(r/q)_4 = +1$.

Hudson and Williams have proved that when the class number of K is not a perfect square there are always solutions of (1) with $p \mid (x^2 - qw^2)$. We call these zero solutions and in this paper we examine the properties of such solutions in some detail (see, particularly, §2).

A major contribution of our paper appears in §4 where we derive explicit formulae for inductively generating all soutions of (1) for $k > 1$ given a basic solution for $k = 1$. Finally in §7 we apply the formulae in §4 to illustrate the Hudson-Williams-Buell extension of a theorem of Cauchy and Jacobi (see [15]).

1. Introduction and summary. Throughout this paper q will denote a positive integer $\equiv 5 \pmod 8$, $q = a^2 + b^2$, with a odd, so that

$$K = Q\left(i\left(\sqrt{2q + 2a\sqrt{q}}\right)\right)$$

is an imaginary cyclic quartic field. For primes $p = qf + 1$, systems of the type

$$(1.2)\qquad \begin{aligned} 16p^k &= x^2 + 2qu^2 + 2qv^2 + qw^2, \\ xw &= av^2 - 2buv - au^2, \qquad (x, u, v, w, p) = 1, \end{aligned}$$