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ON THE KO-ORIENTABILITY OF COMPLEX PROJECTIVE VARIETIES

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The essence of the Riemann-Roch theorem as generalized by P. Baum, W. Fulton, and R. MacPherson is the construction of a natural transformation

$$\alpha_0\colon K_0^{\mathrm{alg}}X \to K_0^{\mathrm{top}}X$$

from the Grothendieck group $K_0^{\mathrm{alg}}X$ of coherent algebraic sheaves on a complex quasi-projective variety X to the topological homology group $K_0^{\mathrm{top}}X$ complementary to the obvious natural transformation

$$\alpha^0\colon K_{\mathrm{alg}}^0X \to K_{\mathrm{top}}^0X$$

from the Grothendieck group K_{alg}^0X of algebraic vector bundles on X to the Atiyah-Hirzebruch group K_{top}^0X of topological vector bundles. Under this natural transformation, the class of the structure sheaf $\mathcal{O}_X$ corresponds to a homology class $\{X\}$,

$$\alpha_0[\mathcal{O}_X] = \{X\},$$

the K-orientation of X. Thus all varieties, singular or non-singular, are K-oriented, in contrast to the well-known fact that a smooth manifold M is K-orientable if and only if the Stiefel-Whitney class $w_3M = 0 \in H^3(M, \mathbf{Z})$.

In this paper we begin the study of the problem of constructing KO-orientations for singular spaces by asking for which varieties X of complex dimension k the class $\{X\}$ lies in the image of the homomorphism

$$\varepsilon_{2k}\colon KO_{2k}X \to K_0X,$$

where

$$\varepsilon.\colon KO.X \to K.X$$

is the natural transformation dual to the complexification homomorphism

$$\varepsilon^{\cdot}\colon KO^{\cdot}X \to K^{\cdot}X$$

from the group of real vector bundles to the group of complex vector bundles. If X is non-singular, then it is necessary and sufficient that the Chern class $c_1X = 0$.