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TOPOLOGICAL METHODS FOR C^*-ALGEBRAS III: AXIOMATIC HOMOLOGY

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A homology theory consists of a sequence $\{h_n\}$ of covariant functors from a suitable category of C^*-algebras to abelian groups which satisfies homotopy and exactness axioms. We show that such theories have Mayer-Vietoris sequences and (if additive) commute with inductive limits. There are analogous definitions and theorems in cohomology with one important difference: an additive cohomology theory associates a Milnor $\lim^1$ sequence to an inductive limit of C^*-algebras. As prerequisite to these results we develop the necessary homotopy theory, including cofibrations and cofibre theories.

0. Introduction.

> The construction of a homology theory is exceedingly complicated. It is true that the definitions and necessary lemmas can be compressed within ten pages, and the main properties established within a hundred. But this is achieved by disregarding numerous problems raised by the construction, and ignoring the problem of computing illustrative examples. ...
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> In spite of this confusion, a picture has gradually evolved of what is and should be a homology theory. Heretofore this has been an imprecise picture which the expert could use in his thinking but not in his exposition. A precise picture is needed. It is at just this stage in the development of other fields of mathematics that an axiomatic treatment appeared and cleared the air.
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> S. Eilenberg and N. Steenrod
> Foundations of Algebraic Topology
> (1952).

There are several homology theories and cohomology theories defined on suitable categories of C^*-algebras. Here are some examples in rough historical order:

(a) The K-theory groups $K_*(A)$ of Karoubi [**7**], [**8**], defined abstractly in terms of modules or concretely via projections and unitaries in matrix algebras over A.

(b) The Ext groups $\mathrm{Ext}^*(A)$ of Brown-Douglas-Fillmore [**3**], [**4**] which arise from the classification of extensions of C^*-algebras of the form

$$0 \to \mathcal{K} \to E \to A \to 0$$