PACIFIC JOURNAL OF MATHEMATICS
Vol. 114, No. 2, 1984

REPRESENTATIONS ASSOCIATED WITH ELLIPTIC SURFACES

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An elliptic surface (over C) $f\colon X \to S$ with a section has two representations naturally associated to it: the first, the monodromy representation, is determined by the topology of f, while the second, the Galois representation, is determined by the arithmetic of the general fiber of f. The purpose of this paper is to study and compare the properties of these representations.

We will always assume that $f\colon X \to S$ is relatively minimal and that the j-invariant is nonconstant. We let K denote the function field of S and E the general fiber of f. Then E/K is an elliptic curve with $f\colon X \to S$ as its Néron model.

The Galois representation given by the action of $\mathrm{Gal}(\bar{K}/K)$ on the torsion points of $E(\bar{K})$ is studied first. Since $\mathbf{C}$ contains all roots of unity, this representation can be regarded as a continuous homomorphism

$$\rho_{E/K}\colon \mathrm{Gal}(\bar{K}/K) \to \mathrm{SL}(2, \hat{\mathbf{Z}}) = \prod_{p\,\text{prime}} \mathrm{SL}(2, \mathbf{Z}_p).$$

With the above hypothesis on E/K, it is known that the image of $\rho_{E/K}$, denoted $\mathrm{Im}(\rho_{E/K})$, is open in $\mathrm{SL}(2, \hat{\mathbf{Z}})$ (see [**5**]). This naturally leads to the notion of *level* of E/K. In §1 we introduce this and study its basic properties. Then, in §2, we show how to bound the level in terms of the behavior of the j-invariant and also in terms of the genus g of K.

The monodromy representation (also called the homological invariant) of $f\colon X \to S$ is studied in §3. If $S_0 = \{s \in S\colon f \text{ is smooth above } s\}$ and X_t is the fiber over $t \in S_0$, then $\pi_1(S_0, t)$ acts on $H^1(X_t, \mathbf{Z})$, giving us the monodromy representation

$$\rho_{X/S}\colon \pi_1(S_0, t) \to \mathrm{SL}(2, \mathbf{Z}).$$

(The image is in $\mathrm{SL}(2, \mathbf{Z})$ because of Poincaré duality.) We will show that the monodromy determines the Galois representation and that in some respects the monodromy is the more subtle invariant.

1. We will work in a slightly more general context than that of the introduction. Here, K will be a field of characteristic zero containing all roots of unity, and E/K will be an elliptic curve such that $\mathrm{Im}(\rho_{E/K})$ is