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A HARNACK ESTIMATE FOR REAL NORMAL SURFACE SINGULARITIES

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According to Harnack's theorem the number of topological components of the real part of a nonsingular projective curve X defined over R is at most $g(X) + 1$, where $g(X)$ is the genus of X. The purpose of the present paper is to present two estimates which can be considered analogs of Harnack's theorem for normal surface singularities defined over R.

1. Introduction. A simple example will suffice to illustrate the type of result which one may expect. Suppose $A \subseteq \mathbf{P}^2(\mathbf{C})$ is a projective plane curve defined over $\mathbf{R}$ and let $A_\mathbf{R}$ be the real part of A. Let $V \subseteq \mathbf{C}^3$ be the cone over A and let $(V_\mathbf{R}, 0)$ be the germ at 0 of the real part of V. Then $(V_\mathbf{R}, 0)$ is connected, but the punctured variety $(V_\mathbf{R} \setminus \{0\}, 0)$ may have two components for each connected component of $A_\mathbf{R}$. Thus the number of components of $(V_\mathbf{R} \setminus \{0\}, 0)$ is bounded by $2 + 2g(A) = b_0(A) + b_1(A) + b_2(A)$ where $b_i(A)$ is the ith betti number of A. If one resolves the singularity $(V, 0)$, the exceptional curve E is just the curve A, so we conclude that the number of components of $(V_\mathbf{R} \setminus \{0\}, 0)$ is bounded by the sum of the betti numbers of the exceptional curve in a resolution of $(V, 0)$. It is in precisely this form that one may obtain a Harnack estimate for an arbitrary normal surface singularity defined over $\mathbf{R}$. Specifically, let (V, p) be a normal surface singularity defined over $\mathbf{R}$ and let $\pi\colon M \to V$ be the minimal normal resolution of V with exceptional curve $E = \pi^{-1}(p)$. Then the following three results will be proved.

1.1. THEOREM. $\pi\colon M \to V$ *is a real resolution, i.e. it is defined over* $\mathbf{R}$.

1.2. THEOREM. $b_0(V_\mathbf{R} \setminus \{0\}, 0) \le \sum_{i=0}^{2} b_i(E)$.

1.3. THEOREM. *By Theorem* 1.1, E *is defined over* $\mathbf{R}$ *and there is the estimate* $b_0(E_R) \le 1 + p_g(E)$ *where* $p_g(E)$ *is the geometric genus of* E.

After recalling some definitions and preliminary results in §2, Theorem 1.1 is proved in §3, while §4 contains the proofs of the two Harnack estimates.