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A CONSTRUCTION OF INNER MAPS PRESERVING THE HAAR MEASURE ON SPHERES

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We show, for $n \geq m$, the existence of non-trivial inner maps f: $B^n \to B^m$ with boundary values f_*: $S^n \to S^m$ such that $f_*^{-1}(A)$ has a positive Haar measure for every Borel subset A of S^m which has a positive Haar measure. Moreover, if $n = m$, the equality $\sigma(f_*^{-1}(A)) = \sigma(A)$ holds, where σ is the Haar measure of S^m.

In this paper $\mathbf{C}^n$ is an n-dimensional complex space with inner product defined by $\langle z^1, z^2\rangle = \sum z_i^1 \bar{z}_i^2$, where $z^j = (z_1^j, z_2^j, \ldots, z_n^j)$ for $j = 1, 2$, and the norm $|z| = \langle z, z\rangle^{1/2}$. Let us introduce some notation:

$$B^n = \{z \in \mathbf{C}^n : |z| < 1\}, \qquad S^n = \partial B^n;$$

let d be the metric on S^n:

$$d(z, z^*) = (1 - \operatorname{Re}\langle z, z^*\rangle)^{1/2} = \frac{1}{\sqrt{2}}|z - z^*| \quad \text{for } z, z^* \in S^n,$$

and finally

$$B(z, r) = \{z^* \in S^n : d(z, z^*) < r\} \quad \text{for } z \in S^n \text{ and } r > 0.$$

For every complex function h: $X \to \mathbf{C}$ we define $Z(h) = h^{-1}(0)$. A holomorphic map f: $B^n \to B^m$ is called inner if

$$f_*(z) = \lim_{r \to 1} f(rz) \in S^m \quad \text{for almost every } z \in S^n$$

with respect to the unique, rotation-invariant Borel measure σ_n on S^n such that $\sigma_n(S^n) = 1$. If a continuous function g: $\bar{B}^n \to \mathbf{C}^m$, defined on the closure of B^n, is holomorphic on B^n, we write $g \in A_m(B^n)$ or $g \in A(B^n)$ when $m = 1$. The theorem stated below is a generalization of the result of Aleksandrov [**1**]. Corollary 1 answers the problem given by Rudin [**3**]. Corollary 4 is a result of Aleksandrov obtained independently by the author.

THEOREM. *Let $n \geq m$ and let $g = (g_1, \ldots, g_m) \in A_m(B^n)$, $h \in A(B^n)$ be maps such that $|g(z)| + |h(z)| \leq 1$ and $h(z) \neq 0$ for some $z \in B^n$. Then there exists an inner map $f = (f_1, f_2, \ldots, f_m)$: $B^n \to B^m$ such that $f(z) = g(z)$ for every $z \in Z(h)$ and $f_i(z) = g_i(z)$ for every $z \in B^n$ and $i = 1, 2, \ldots, m - 1$.*