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CLASS NUMBERS OF IMAGINARY CYCLIC QUARTIC FIELDS AND RELATED QUATERNARY SYSTEMS

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A proof is given of an explicit Dirichlet-type class number formula for imaginary cyclic quartic fields obtained in 1980 by Hudson and Williams and, in a slightly different form, by Setzer. The Hudson-Williams formula is used to study the solvability of the quaternary quadratic form

$$16p^k = x^2 + 2qu^2 + 2qv^2 + qw^2,$$
$$xw = av^2 - 2buv - au^2, \qquad (x, u, v, w, p) = 1$$

for exponents $k \geq 1$. Included is a table from which every class number $h(k)$ of the quartic field $k = Q(i\sqrt{2q + 2a\sqrt{q}})$, $q \equiv 5 \pmod 8$ a prime, may be determined for $q < 10000$. Finally, a quartic analog of the well-known result that the number of quadratic residues in $(0, p/2)$ exceeds the number in $(p/2, p)$ if $p \equiv 3 \pmod 4$ is proven using one of Dirichlet's less well-known class number formulas.

1. Introduction. Explicit Dirichlet-type class number formulas for imaginary cyclic quartic fields were obtained in 1980 by Hudson and Williams and independently by Setzer. In this paper we sketch in §2 the proof of the Hudson-Williams formula and show that these two formulas are easy consequences of one another. However, the Hudson-Williams formulation is particularly useful for studying the solvability of the quaternary quadratic form

$$\begin{aligned} &16p^k = x^2 + 2qu^2 + 2qv^2 + qw^2, \\ &xw = av^2 - 2buv - au^2, \qquad (x, u, v, w, p) = 1, \end{aligned} \tag{1.1}$$

for $k \geq 1$. We show in §§3–5 that solvability of this form depends heavily on the relative class number $h^* = h(k)/h(Q(\sqrt{q}))$ of the imaginary cyclic quartic field

$$K = Q\left(i\sqrt{2q + 2a\sqrt{q}}\right) = Q\left(i\sqrt{2q - 2a\sqrt{q}}\right) \tag{1.2}$$

where $q \equiv 5 \pmod 8) = a^2 + b^2$ (a odd, $b > 0$) will denote a prime > 5 throughout and $h(Q(\sqrt{q}))$ the class number of the unique quadratic subfield $Q(\sqrt{q})$ of K.