PACIFIC JOURNAL OF MATHEMATICS
Vol. 115, No 1, 1984

A NOTE ON PROJECTIONS OF REAL ALGEBRAIC VARIETIES

C. ANDRADAS AND J. M. GAMBOA

We prove that any regularly closed semialgebraic set of R^n, where R is any real closed field and regularly closed means that it is the closure of its interior, is the projection under a finite map of an irreducible algebraic variety in some R^{n+k}. We apply this result to show that any clopen subset of the space of orders of the field of rational functions $K = R(X_1,\ldots,X_n)$ is the image of the space of orders of a finite extension of K.

1. Introduction. Motzkin shows in **[M]** that every semialgebraic subset of R^n, R an arbitrary real closed field, is the projection of an algebraic set of R^{n+1}. However, this algebraic set is in general reducible, and we ask whether it can be found irreducible.

This turns out to be closely related to the following problem, proposed in **[E-L-W]**: let $K = R(X_1,\ldots,X_n)$, $X_1,\ldots,X_n$ indeterminates, and let X_K be the space of orders of K with Harrison's topology. If $E|K$ is an ordered extension of K, let $\varepsilon_{E|K}$ be the restriction map between the space of orders, $\varepsilon_{E|K}\colon X_E \to X_K\colon P \mapsto P \cap K$. Which clopen subsets of X_K, that is, closed and open in Harrison's topology, are images of $\varepsilon_{E|K}$ for suitable finite extension of K?.

In this note we prove that every regularly closed semialgebraic subset $S \subset R^n$ — S is the closure in the order topology of its inner points — is the projection of an irreducible algebraic set of R^{n+k} for some $k \geq 1$. Actually we prove more: the central locus of the algebraic set, i.e., the closure of its regular points, covers the whole semialgebraic S. This allows us to prove that there exists an irreducible hypersurface in R^{n+1} whose central locus projects onto S. As a consequence we prove that for every clopen subset $Y \subset X_K$ there is a finite extension E of K such that $\operatorname{im}(\varepsilon_{E|K}) = Y$.

2. In what follows R will be a real closed field and π will always denote the canonical projection of some R^{n+k} onto the first n coordinates.

Let S be a semialgebraic closed subset of R^n. Then S can be written in the form (cf. **[C-C]** **[R]**):

$$S = \bigcup_{i=1}^{p} \{x \in R^n : f_{i1}(x) \geq 0,\ldots,f_{ir}(x) \geq 0\}, \qquad f_{ij} \in R[X_1,\ldots,X_n].$$