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ONE-DIMENSIONAL PERTURBATIONS OF OPERATORS

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The purpose of this note is three-fold. First, to show that certain rank one operators of small norm will split off a unitary piece from the shift; second, to apply this technique to the "Berg construction"; and, third, to exhibit a normal operator whose eigenspace structure is drastically altered by the addition of a rank one operator.

Throughout this paper $\mathscr{L}(\mathscr{H})$ will denote the algebra of bounded linear operators on a separable Hilbert space $\mathscr{H}$.

1. We begin with a proposition which tells us which rank one operators preserve an isometry.

PROPOSITION 1. *Let S be any isometry. Let $F = a(\cdot, g)h$ where $\|g\| = \|h\| = 1$. Then $S + F$ is isometric if and only if*

$$(1) \qquad \begin{cases} S\colon g \to e^{-i\theta}h \\ \\ S^*\colon h \to e^{i\theta}g \end{cases} \qquad \textit{for some } \theta \textit{ real}$$

and

$$(2) \qquad 2\,\mathrm{Re}\, e^{-i\theta}a = -|a|^2.$$

Proof. To show $S + F$ is isometric we must show

$$I = (S + F)^*(S + F) = I + S^*F + F^*S + F^*F$$

or equivalently $S^*F + F^*S + F^*F = 0$. Note that $F^* = \bar{a}(\cdot, h)g$. If $f \in \mathscr{H}$ then

(a) $S^*Ff = ae^{i\theta}(f, g)g$,
(b) $F^*Sf = \bar{a}e^{-i\theta}(f, g)g$, and
(c) $F^*Ff = |a|^2(f, g)g$.

Thus

$$(S^*F + F^*S + F^*F)f = \left[(ae^{i\theta} + \bar{a}e^{-i\theta}) + |a|^2\right](f, g)g = 0$$

if (2) holds.