PACIFIC JOURNAL OF MATHEMATICS
Vol. 115, No. 2, 1984

WHEN THE CONTINUUM HAS COFINALITY ω_1

ARNOLD W. MILLER AND KAREL PRIKRY

In this paper we consider models of set theory in which the continuum has cofinality ω_1. We show that it is consistent with $\neg$CH that for any complete boolean algebra B of cardinality less than or equal to c (continuum) there exists an ω_1-generated ideal J in $P(\omega)$ (power set of ω) such that B is isomorphic to $P(\omega) \bmod J$. We also show that the existence of generalized Luzin sets for every ω_1-saturated ideal in the Borel sets does not imply Martin's axiom.

Introduction. In §1 we prove our main result that it is consistent with $\neg$CH that every complete boolean algebra of cardinality $\leq c$ is isomorphic to $P(\omega) \bmod J$ for some J ω_1-generated. We think of this as generalizing Kunen's theorem that it is consistent with $\neg$CH that there is an ω_1 generated nonprincipal ultrafilter on ω.

For I an ideal in the Borel subsets of the reals we say that a set of reals X is a κ-I-Luzin set iff X has cardinality κ and for every A in I, $A \cap X$ has cardinality less than κ. If c is regular, then it follows easily from Martin-Solovay [**9**] that MA is equivalent to the statement "for every ω_1-saturated σ-ideal I in the Borels there is a c-I-Luzin set". In §2 we show that the regularity of c is necessary. This answers a question of Fremlin [**5**].

We also show that it is consistent with $\neg$CH that for every such I there exists an ω_1-I-Luzin set. These results can be thought of as a weak form of the following conjecture.

Conjecture. It is consistent with $\neg$CH that for every c.c.c partial order $\mathbf{P}$ of cardinality $\leq c$ there exist $\langle G_\alpha: \alpha < \omega_1\rangle$ an ω_1-sequence of $\mathbf{P}$-filters such that for every dense $D \subseteq \mathbf{P}$ all but countably many G_α meet D.

Note that this is a trivial consequence of CH.

Next we give a result of Kunen that some restriction of the cardinality of $\mathbf{P}$ (e.g. $(2^{\omega_1})^+$) is necessary in our conjecture. We also show that for every c.c.c. $\mathbf{P}$ of cardinality $\leq \omega_2$ we can force (without adding reals) the existence of $\mathbf{P}$-filters $\langle G_\alpha: \alpha < \omega_1\rangle$ eventually meeting each dense subset of $\mathbf{P}$.