PACIFIC JOURNAL OF MATHEMATICS
Vol. 116, No. 1, 1985

INTEGRALITY OF SUBRINGS OF MATRIX RINGS

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Let $A \subseteq B$ be commutative rings, and Γ a multiplicative monoid which generates the matrix ring $M_n(B)$ as a B-module. Suppose that for each $\gamma \in \Gamma$ its trace $\operatorname{tr}(\gamma)$ is integral over A. We will show that if A is an algebra over the rational numbers or if for every prime ideal P of A, the integral closure of A/P is completely integrally closed, then the algebra $A(\Gamma)$ generated by Γ over A is integral over A. This generalizes a theorem of Bass which says that if A is Noetherian (and the trace condition holds), then $A(\Gamma)$ is a finitely generated A-module.

Our generalizations of the theorem of Bass [**B**, Th. 3.3] yield a simplified proof of that theorem. Bass's proof used techniques of Procesi in [**P**, Ch. VI] and involved completion and faithfully flat descent. The arguments given here are based on elementary properties of integral closure and complete integral closure. They serve also to illuminate a couple of theorems of A. Braun concerning prime p.i. rings integral over the center.

One might expect that integrality of $\operatorname{tr}(\gamma)$ for $\gamma \in \Gamma$ would be sufficient to assure that $A(\Gamma)$ is integral over A. But this is not so, as we will show with a counterexample. As it frequently happens with traces, complications arise in prime characteristic.

1. Integrality and complete integral closure. Recall that if A is an integral domain and b lies in its quotient field, b is said to be *almost integral* over A if there is an $a \in A$, $a \neq 0$, such that $ab^i \in A$ for all integers $i \geq 1$. A is said to be *completely integrally closed* (c.i.c.) if every element almost integral over A lies in A. Recall that a Krull domain is completely integrally closed [**Bo**, §1, No. 3], as indeed is any intersection of rank 1 valuation rings. (However, examples are known of c.i.c. domains which are not intersections of rank 1 valuation rings — see [**Nk**] or [**G**, App. 4].) If A is a Noetherian domain, the Mori-Nagata Theorem [**N**, (33.10)] says that the integral closure of A is a Krull domain, hence is c.i.c.

LEMMA 1. *Let A be a completely integrally closed integral domain with quotient field F, and let B be the integral closure of A in any extension field of F. Then B is completely integrally closed.*