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NONASSOCIATIVE ALGEBRAS WITH SCALAR INVOLUTION

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The classical theory of nondegenerate quadratic forms permitting composition has recently been generalized in several directions: Kunze and Scheinberg considered degenerate forms on alternative algebras over fields of characteristic $\neq 2$; Petersson and Racine briefly considered nondegenerate forms over general rings of scalars; the generalized Cayley-Dickson algebras of dimension 2^n carry a scalar involution, but are not alternative and do not admit composition for $n > 3$. In this paper we study general algebras with scalar involution (where all norms xx^* and traces $x + x^*$ are scalars) over arbitrary rings of scalars. We locate these among all degree 2 algebras, and derive conditions for them to be flexible, alternative, or composition algebras. We consider Cayley elements and Cayley birepresentations, recovering the results of Kunze and Scheinberg on radicals of norm forms. We also investigate the Cayley-Dickson doubling process for constructing new scalar involutions out of old ones.

Throughout, A denotes a unital nonassociative algebra over an arbitrary (unital, commutative, associative) ring of scalars Φ. On occasion pathologies in the module structure of A will cause problems. Without loss of generality (replacing Φ by $\Phi/A^{\perp}$) we will always assume that Φ acts *faithfully* on A,

$$\alpha A = 0 \Rightarrow \alpha = 0 \tag{0.1}$$

or equivalently that Φ is *unitally faithful*,

$$\alpha 1 = 0 \Rightarrow \alpha = 0 \tag{0.1$'$}$$

(since if α kills the unit it kills all of A). In order to insure uniqueness of traces and norms we will sometimes impose a stronger condition (unnecessary for free modules or for Φ without nilpotent elements) of "unital rigidity" (cf. §2).

As usual, an *involution* is an anti-automorphism of period 2,

$$(xy)^* = y^*x^*, \qquad x^{**} = x. \tag{0.2}$$

A *scalar involution* is one for which all *norms* xx^* are scalars in $\Phi 1$, hence by linearization all *traces* $x^* + x^*$ are too; by faithfulness (0.1′) these