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# REMARKS ON THE PAPER 'BASIC CALCULUS OF VARIATIONS'

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**We show that a condition studied in E. Silverman's paper is not, as claimed, necessary for lower semicontinuity of multiple integrals in the calculus of variations.**

The purpose of this note is to show that a condition studied in [7] is not, as claimed, a necessary condition for lower semicontinuity of multiple integrals in the calculus of variations. To keep things simple we consider integrals of the form

$$I_F(y) = \int_G F(y'(x))\,dx,$$

where $G \subset \mathbf{R}^k$ is a bounded domain, $y: G \to \mathbf{R}^N$, $y'(x) = (\partial y^i/\partial x^\alpha)$, and $F: M^{N\times k} \to \mathbf{R}$ is continuous. Here $M^{N\times k}$ denotes the linear space of real $N \times k$ matrices. We suppose throughout that $K \geq 2$, $N \geq 2$. In [7] $F$ is called *T-convex* if there exists a convex function $f$, defined on $\mathbf{R}^r$, $r = \binom{N+k}{k} - 1$, such that

$$F(p) = f(\tau(p)) \quad \text{for all } p \in M^{N\times k},$$

where $\tau(p)$ denotes the minors of $p$ of all orders $j$, $1 \leq j \leq \min(k, N)$, arranged in some prescribed order. $T$-convexity of $F$ was studied in [**1**, **2**, **3**] under a different name, *polyconvexity*, which we shall use in the remainder of this note, and it is equivalent to a condition introduced earlier by Morrey [**4**, p. 49]. (These papers contain lower semicontinuity and existence theorems for polyconvex integrands of the same type as given in [**7**, §§4–7].) Let us say that $I_F$ is lsc if $I_F(y) \leq \liminf_{j\to\infty} I_F(y_j)$ whenever $y_j \to y$ uniformly on $G$ with $\sup_{x,\bar{x}\in G}|y_j(x) - y_j(\bar{x})| \leq C < \infty$ for all $j$. (Equivalently, if $G$ has sufficiently regular boundary then $I_F$ is lsc if and only if $I_F$ is sequentially weak* lower semicontinuous on the Sobolev space $W^{1,\infty}(G; \mathbf{R}^n)$.) A consequence of [**7**, Theorem 3.6] is that $I_F$ lsc implies $F$ polyconvex; that this conclusion is false was pointed out implicitly by Morrey [**4**, p. 26]. Morrey's remark is based on an example due to Terpstra [**8**] of a quadratic form

$$Q(p) = \sum_{\substack{1\leq i,j\leq N \\ 1\leq \alpha,\beta\leq k}} a_{i\alpha j\beta}\, p_{i\alpha} p_{j\beta}$$