PACIFIC JOURNAL OF MATHEMATICS
Vol. 116, No. 2, 1985

AN ANALOGUE OF LIAPOUNOFF'S CONVEXITY THEOREM FOR BIRNBAUM-ORLICZ SPACES AND THE EXTREME POINTS OF THEIR UNIT BALLS

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To Z. W. Birnbaum, on the occasion of his 80th birthday

Let (X, S, μ) be a non-atomic probability space. Our purpose is to note an analogue of Liapounoff's convexity theorem, as a statement about $L^{\infty}(\mu)$, for certain real Birnbaum-Orlicz spaces $L_{\Phi}(\mu)$, in particular reflexive ones, under the usual norms: the extreme points of the unit ball yield the full image of the ball under finite dimensional continuous linear maps.

1. Of course such a result is trivial if the ball is strictly convex (when each of its boundary points is extreme) since it just asserts that any finite codimensional closed subspace which meets the ball meets its extreme elements. But for $L_{\Phi}(\mu)$ the unit ball will have flat spots on its boundary corresponding to horizontal segments in the graph of $\varphi = \Phi'$, and the strong sort of density of extreme points the result implies is nontrivial. We shall obtain the analogue in fact as an application of Liapounoff's theorem (resulting from the use of support functionals suggested by **[3]**), and, although characterizations of the extreme points of the balls could be avoided, we shall obtain these too, so that Lindenstrauss' elegant proof of the Liapounoff result **[5, 6]** can also be applied.

Needless to say the assertion of the result makes sense for any Banach space, and fails if no extreme points exist; indeed it fails for $L^{\infty}(\mu)$ if our map is not w^* continuous. But it easily fails with that restriction, for example for the space of real measures on $[-2, -1] \cup [1, 2]$ and the map into $\mathbf{R}$ provided by $\nu \to \int \operatorname{sgn} t\, \nu(dt)$.

Finally we adapt the argument of **[3]** to one instance where neither of the approaches to the reflexive case applies (Theorem 2 below).

2. Let Φ and Ψ be dual Young's functions **[2, 7]** i.e., $\Phi(x) = \int_0^x \varphi(t)\, dt$, $\Psi(x) = \int_0^x \psi(t)\, dt$, where φ: $\mathbf{R}_+ \to \mathbf{R}_+$ is 0 at 0, non-decreasing and $\to \infty$ at ∞, while ψ is its "inverse", both taken continuous from the left for definiteness. As in **[7]** we complete the graph of φ by adding vertical segments at any discontinuities, and speak of the resulting curve