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TOPOLOGIES ON THE QUOTIENT FIELD OF A DEDEKIND DOMAIN

JO-ANN COHEN

It is well known that if D is a Dedekind domain with quotient field K and if T is any Hausdorff nondiscrete field topology on K for which the open D-submodules of K form a fundamental system of neighborhoods of zero, then T is the supremum of a family of p-adic topologies. We show that if the class number of K over D is finite and if T is any Hausdorff nondiscrete field topology on K for which D is a bounded set, then T is the supremum of a family of p-adic topologies. We then investigate the problem of extending a locally bounded topology from D to a locally bounded topology on K. The extendable topologies on D for which there exists a nonzero topological nilpotent and for which D is a bounded set are characterized. Moreover it is shown that the topology of a locally compact principal ideal domain A extends to a ring topology on the quotient field of A if and only if A is compact.

1. Introduction and basic definitions. Let R be a commutative ring and let T be a ring topology on R, that is, T is a topology on R making $(x, y) \to x - y$ and $(x, y) \to xy$ continuous from $R \times R$ to R. A subset A of R is *bounded* for T if given any neighborhood U of zero, there exists a neighborhood V of zero such that $AV \subseteq U$. T is a *locally bounded topology* on R if there exists a fundamental system of neighborhoods of zero for T consisting of bounded sets. As every compact set is bounded [**4**, Exercise 12, p. 119], if T is a ring topology on R and (R, T) is locally compact, then T is a locally bounded topology on R.

Each norm N on a ring R defines a locally bounded topology T_N on R in a natural way. Obviously, each norm-bounded subset of R is also bounded for T_N. Furthermore, if N is a nontrivial norm on a field K, that is, T_N is nondiscrete, then a subset A of K is bounded in norm if and only if A is bounded for T_N.

Let D be a Dedekind domain that is not a field, let K be the quotient field of D and let $\mathscr{P}$ be the set of nonzero proper prime ideals of D. We assume familiarity with the definitions and basic properties of the functions n_p defined on the set of nonzero fractionary ideals of K and the valuations v_p defined on K for each p in $\mathscr{P}$. (See for example [**3**, pp. 25–26].) If F is a field and x is a transcendental element over F, we denote the valuation on $F(x)$ defined by the prime ideal (x^{-1}) of $F[x^{-1}]$ by v_∞.