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THE GENERALIZED SCHWARZ LEMMA FOR THE BERGMAN METRIC

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The function-theoretic criterion for the Bergman metric to be dominated by the Kobayashi metric on the domain in $\mathbf{C}^n$ is given. For this, we use the distinguished family of plurisubharmonic functions and P-metric of N. Sibony.

1. Introduction. Let D be a hyperbolic domain in $\mathbf{C}^n$ (cf. Kobayashi [7]). On D we can define some intrinsic metrics: the Carathéodory metric C_D, the Kobayashi metric K_D, and the Bergman metric B_D. It is known that $C_D \le K_D$ and $C_D \le B_D$. In this paper we investigate when the Bergman metric is dominated by the Kobayashi metric. Using N. Sibony's P-metric we give a function-theoretic criterion for the following condition (#) to hold:

(#) $B_D \le cK_D$ on the tangent bundle of D, $c > 0$ constant.

Under (#) every holomorphic mapping $F\colon U \to D$ satisfies $F^*B_D \le 2^{-1/2}cB_U$, where U is the unit disc in $\mathbf{C}$ with the Bergman metric B_U. This theorem is called the generalized Schwarz lemma for the Bergman metric. According to N. Sibony [**10**], we introduce the family of functions

$$S_p(D) = \{u\colon D \to [0,1);\ u(p) = 0,\ C^2\text{-class in a neighborhood of } p \text{ and } \log u \text{ is plurisubharmonic in } D\}.$$

Taking a Bergman kernel $k(z, \bar{w})$ of a domain D, we construct a function ϕ_w for a fixed point w in D as follows;

$$\phi_w(z) = \phi_{w,\alpha}(z) = 1 - \left(\frac{|k(z,\bar{w})|^2}{k(z,\bar{z})k(w,\bar{w})}\right)^{\alpha} \equiv 1 - v^{\alpha},$$

where α is a positive constant chosen for D. It is clear that $0 \le \phi_w \le 1$ and $\phi_w(w) = 0$. Our main results are stated as follows.

(I) Let D be a Bergman domain. If there is a constant $\alpha > 0$ such that $\phi_w = \phi_{w,\alpha}$ belongs to $S_w(D)$ for each w in D, then $B_D \le \alpha^{-1/2}K_D$; hence B_D satisfies the generalized Schwarz lemma.