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NORM-ATTAINMENT OF LINEAR FUNCTIONALS ON SUBSPACES AND CHARACTERIZATIONS OF TAUBERIAN OPERATORS

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It is proved that for every non-zero continuous linear functional f on a non-reflexive Banach space X, there is a closed linear subspace Y so that $f\,|_Y$ does not attain its norm. In fact, Y may be chosen with $\|f\,|_Y\|$ arbitrarily close to $\|f\|$. It is also shown that every continuous linear functional on an infinite-dimensional normed linear space fails to attain its norm on some linear subspace. The class of non-zero Banach space operators which map closed bounded convex sets to closed sets is identified as the class of Tauberian operators. (A bounded linear operator T: $X \to Y$ is defined to be Tauberian provided $T^{**}x^{**} \in Y$ implies $x^{**} \in X$.) Other closed image characterizations are obtained. In particular, using the very first result stated above, a non-zero operator is found to be Tauberian if and only if the image of the ball of any closed subspace is closed. The new characterizations show that the "hereditary versions" of semi-embeddings and F_σ-embeddings are precisely the one-to-one Tauberian operators.

Introduction. Let X and Y be Banach spaces and T: $X \to Y$ a given non-zero operator. (Throughout, "operator" means "bounded linear map".) Under what circumstances is it true that TK is closed for every closed bounded convex subset K of X? Evidently this is trivially true if X is reflexive, so suppose this is not the case. Here are some of the equivalences obtained in our main result of §2, Theorem 2.3. (For any Banach space Z, let $B_Z = \{z \in Z: \|z\| \leq 1\}$.)

THEOREM. *Assume T: $X \to Y$ is non-zero with X non-reflexive. The following are equivalent*:

(a) *T is Tauberian.*

(b) *TK is closed for all closed bounded convex K.*

(c) *TB_Z is closed for all closed linear subspaces Z.*

(d) *TX is infinite-dimensional and TZ is an F_σ for all closed linear subspaces Z.*

We recall the definition that T: $X \to Y$ is *Tauberian* provided whenever $G \in X^{**}$ and $T^{**}G \in Y$, then $G \in X$ (where we regard X as canonically embedded in X^{**}). Results of Kalton and Wilansky concerning Tauberian