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REMARKS ON THE DEDEKIND COMPLETION OF A NONSTANDARD MODEL OF THE REALS

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In 1980 Wattenberg constructed the Dedekind completion of a nonstandard model of the real numbers and applied the construction to obtain certain kinds of special measures on the set of integers. We feel that the Dedekind completion is a structure of interest for its own sake and we establish further properties here. Of particular interest is the connection with [1]. Specifically, the main concept we introduce is that of the absorption number of an element a which, roughly speaking, measures the degree to which the cancellation law $a + b = a + c \to b = c$ fails for a. The absorption number may be regarded as an element in the Dedekind completion of the value group of the valuation ring of finite numbers as discussed in [1].

Preliminaries. Let R be the set of real numbers and R^* a nonstandard model of R. Also let $R^\#$ be the class of all lower subsets α of R^* which are non-empty, with non-empty complement and with no greatest element. $R^\#$ is the Dedekind completion of R^* according to [**2**, page 227]. We identify $a \in R^*$ with $\alpha = \{x: x < a\}$.

DEFINITION. $\alpha + \beta = [a + b: a \in \alpha \wedge b \in \beta]$.

Addition is commutative and associative. Furthermore, $\alpha + 0 = \alpha$. Also, the embedding $R^* \to R^\#$ preserves sums.

DEFINITION. $-\alpha = (a \in R^*: \exists b[b > a \wedge -b \notin \alpha])$[1].

As we already noted the embedding $R^* \to R^\#$ preserves negation.

DEFINITION. $\alpha \leq \beta$ iff $\alpha \subset \beta$.

Then clearly $\beta \leq \gamma \to \alpha + \beta \leq \alpha + \gamma$.

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[1]Note that the definition in [**2**] is technically incorrect. The latter defines $-\alpha$ as $(a \in R^*: -a \notin \alpha)$. If $\alpha \notin R^*$ i.e. α' has no mininum, then the definitions are equivalent. However if $\alpha \in R^*$ i.e. $\alpha = (x: x < a)$ for some $a \in R^*$ then according to the latter definition $-\alpha = (x: x \leq -a)$ whereas the definition of $R^\#$ requires that $-\alpha = (x: x < -a)$. In the usual treatment of Dedekind cuts for the ordinary real numbers both of the latter sets are regarded as equivalent so that no serious problem arises.