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ON SPECIAL PRIMES

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In fond memory of Ernst Straus

A special prime q is a prime which divides the discriminant of a general period polynomial of degree e associated with the prime $p = ef + 1$, but q is neither an eth power residue of p nor a divisor of any value of this polynomial.

These primes are very rare. Evans found some for the classical cyclotomic octic. There are none for lower degree cyclotomic polynomials. This paper finds special primes for the two quartics arising from the cyclotomy of Kloosterman sums for $e = 8$ and shows that there are none for $e < 8$.

Introduction. In two earlier papers on the cyclotomy of Kloosterman Sums [**3, 4**] we proved that for e a prime and $p = ef + 1$ the Kloosterman equation is irreducible and represents numbers all of whose prime factors are eth power residues of p. However, when e is even the equation splits into two irreducible equations of degree $e/2$. The question arises as to whether these two equations can have factors which are not $e/2$th power residues. Such factors are called *exceptional* and they have to divide the discriminant of the equations. Evans [**2**] raised the question of whether all the divisors of the discriminant of the cyclotomic period polynomials are eth power residues. He called the divisors of he discriminant which are not eth power residues *semiexceptional* and showed that for $e = 8$ there exist semiexceptional divisors which are not exceptional. We studied the problem for $e = 6$ in [**5**], where we called such semiexceptional divisors *special*, and showed that they do not exist for $e = 6$. In [**8**] I studied a wider range of period equations of degree $2e$, where e is a prime without finding any special primes. In what follows such primes will be found for the two Kloosterman quartics for $e = 8$ together with the exceptional primes for the two cubics for $e = 6$, as well as for the two quadratics found in [**3**] for $e = 4$.

1. Kloosterman sums. The Kloosterman sum $S(h)$ is defined by

$$S(h) = \sum_{x=1}^{p-1} \epsilon(x + h\bar{x}), \qquad x\bar{x} \equiv 1 \pmod{p}$$