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THE CHROMATIC POLYNOMIAL OF A GRAPH

D. H. LEHMER

Dedicated to the memory of Ernst G. Straus

The problem of which polynomials occur as the vertex coloring of a graph is studied. The results are complete for graphs with fewer than seven vertices.

The vertex coloring of a graph is an assignment of colors to the vertices of the graph in such a way that no edge has its end points colored the same. If λ colors are available the number of ways that a graph G can be so colored is denoted by $P(G, \lambda)$. This function is a monic polynomial of degree V with integer coefficients, where V is the number of vertices. The smallest positive integer λ for which $P(G, \lambda) > 0$ is called the chromatic number of G. For example, the chromatic polynomial of a triangle is easily seen to be

$$\lambda^3 - 3\lambda^2 + 2\lambda = \lambda(\lambda - 1)(\lambda - 2)$$

while that for the cube is

$$\begin{aligned}&\lambda^8 - 12\lambda^7 + 66\lambda^6 - 214\lambda^5 + 441\lambda^4 - 572\lambda^3 + 423\lambda^2 - 133\lambda \\ &\quad = \lambda(\lambda - 1)(\lambda^6 - 11\lambda^5 + 55\lambda^4 - 159\lambda^3 + 282\lambda^2 - 290\lambda + 133).\end{aligned}$$

Thus the cube can be colored in two colors in just two ways while the triangle requires three colors and then can be colored in six ways.

An outstanding problem in graph theory is to decide whether a given polynomial is the chromatic polynomial of some graph. It is the purpose of this paper to look into this question. A complete account is given for graphs with fewer than seven vertices. A statistical sample of 100 of the graphs with 10 vertices is also discussed.

Suppose the given polynomial is

$$(1) \quad P(\lambda) = \lambda^v - a_1\lambda^{v-1} + a_2\lambda^{v-2} + \cdots + (-1)^{v-1}a_{v-1}\lambda + (-1)^v a_v.$$

Then v must be the number of vertices of the proposed graph G. Secondly it is known that a_1 is the number of edges of G. Furthermore, it is known that $a_i \geq 0$. Since a graph with at least one edge cannot be colored in less than two colors, it follows that $P(\lambda)$ is divisible by $\lambda(\lambda - 1)$. In particular