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THE SET OF PRIMES DIVIDING THE LUCAS NUMBERS HAS DENSITY 2 / 3

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Dedicated to the memory of Ernst Straus

The Lucas numbers L_n are defined by $L_0 = 2$, $L_1 = 1$ and the recurrence $L_n = L_{n-1} + L_{n-2}$. The set of primes $S_L = \{p: p$ divides L_n for some $n\}$ has density 2/3. Similar density results are proved for sets of primes $S_U = \{p: p$ divides U_n for some $n\}$ for certain other special second-order linear recurrences $\{U_n\}$. The proofs use a method of Hasse.

1. Introduction. There has been a good deal of study of the structure of the set of prime divisors of the terms $\{U_n\}$ of second order linear recurrences. M. Ward [**15**] showed that there are always an infinite number of distinct primes dividing the terms $\{U_n\}$, provided we exclude certain degenerate cases such as $U_n = 2^n$. In fact, under the same circumstances it is believed that the set of primes dividing the terms $U = \{U_n\}$ of any nondegenerate second order linear recurrence has a positive density $d(U)$ depending on the recurrence. This can be proved under the assumption that the Generalized Riemann Hypothesis is true by a method analogous to Hooley's conditional proof [**4**] of Artin's Conjecture for primitive roots. P. J. Stephens [**13**] has done this for a large class of second-order linear recurrences.

The point of this paper is that there are special second order linear recurrences where it is possible to give an unconditional proof of the existence of a density. This was shown by Hasse [**3**] for certain special second order linear recurrences having a reducible characteristic polynomial, in the process of solving a problem of Sierpinski [**12**]. Sierpinski's problem concerns the existence of a density for the set of primes p for which $\text{ord}_p 2$ is even. This set of primes is exactly the set of primes dividing some term of the sequence $V_n = 2^n + 1$; this sequence satisfies the reducible second order linear recurrence $V_n = 3V_{n-1} - 2V_{n-2}$ with $V_0 = 2$ and $V_1 = 3$.

THEOREM A. (*Hasse*) *The set of primes $S_V = \{p$: p is prime and p divides $2^n + 1$ for some $n \geq 0\}$ has density* 17/24.