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THE RADON TRANSFORM ON $\mathbf{Z}_2^k$

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In memory of Ernst Straus

Suppose G is a finite group and f is a function mapping G into the set of real numbers $\mathbf{R}$. For a subset $S \subseteq G$, define the Radon transform F_S of f mapping G into $\mathbf{R}$ by:

$$F_S(x) = \sum_{y \in S+x} f(y)$$

where $S + x$ denotes the set $\{s + x: s \in S\}$. Thus, the Radon transform can be thought of as a way of replacing f by a "smeared out" version of f. This form of the transform represents a simplified model of the kind of averaging which occurs in certain applied settings, such as various types of tomography and recent statistical averaging techniques.

A fundamental question which arises in connection with the Radon transform is whether or not it is possible to invert it, i.e., whether one can recover (in principle) the function f from knowledge of F_S.

In this paper we investigate this problem in detail for several special classes of groups, including the group of binary n-tuples under modulo 2 addition.

1. Introduction. Let X be a finite set and let Y be a class of subsets of X. For a real-valued function f: $X \to \mathbf{R}$, the Radon transform of f at $y \in Y$ is defined as

$$\bar{f}(y) = \sum_{x \in y} f(x).$$

This paper investigates uniqueness of the transform when X is the group of binary k-tuples $\mathbf{Z}_2^k$ or the symmetric group S_n, and Y is the class of translates of a given set $S \subset X$.

In §2 we deal with $\mathbf{Z}_2^k$. It is shown that the transform is one-to-one when $|S|$ is odd and is not one-to-one for most sets of even cardinality. It is also shown that the problem of determining uniqueness is NP-complete so that at present no polynomial-time algorithm (in k and $|S|$) is known to exist to determine uniqueness.

In Section 3 we give explicit inversion theorems for the case where $S = \{x \in \mathbf{Z}_2^k: H(0, x) \leq 1\}$ where $H(x, y)$ is Hamming distance—the number of coordinates where x and y disagree. The transform is one-to-one