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COMPOSITION ALGEBRAS OF POLYNOMIALS

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Dedicated to the memory of Ernst G. Straus

Briefly, a composition algebra A involves two operations: addition and composition (substitution of polynomials). Let C be an arbitrary commutative ring, and $C[x, y, \ldots]$ the ring of polynomials in the indeterminates $x, y, \ldots$ with coefficients from C. Addition of polynomials is commutative; composition is associative, and is distributive (on one side) over addition. (Notice that if the number of indeterminates is greater than 1, the operation of composition is not a binary operation.) We find the ideal structure of A in some special cases. In particular, the ideals of A are all principal (generated by a single element) if C is a principal ideal ring (e.g. Z) and the number of variables is 1: $A = (C[x], +, \circ)$, provided further that for all $c \in C$, $2|c + c^2$. [An example is the algebraic integers in $Q(\sqrt{-7})$.]

We start in a general context. An ideal J in A is the kernel of a homomorphism. Thus J enjoys these three properties:

1.01. J is a module over C: If $c_1, c_2 \in C$, $t_1, t_2 \in J$, then $c_1t_1 + c_2t_2 \in J$.

1.02. If $t \in J$ and $n_1, n_2, \ldots \in A$, then $t(x, y, \ldots) \circ [n_1, n_2, \ldots] \equiv t(n_1, n_2, \ldots)$ lies in J.

1.03. If $t_2, t_3, \ldots \in J$ and if $n_1, n_2, \ldots \in A$, then $n_1(x, y, \ldots) \circ [n_2 + t_2, n_3 + t_3, \ldots] - (n_1(x, y, \ldots) \circ [n_2, n_3, \ldots])$ lies in J.

Since n_1 is a sum of monomials, it follows from 1.01 that 1.03 can be replaced by the simpler requirement

1.04. $\prod_{i=2}^{k}(n_i + t_i)^{\alpha_i} - \prod_{i=2}^{k} n_i^{\alpha_i}$ lies in J.

1.05. DEFINITION. An ideal $J = \langle a \rangle$ in A is *principal* if J is the smallest ideal containing a. A is a *principal ideal composition algebra* (in short, A is principal) if every ideal is principal.