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PATH PARTITIONS AND PACKS OF ACYCLIC DIGRAPHS

R. AHARONI, I. BEN-ARROYO HARTMAN AND A. J. HOFFMAN

In memory of Ernst Straus

Let G be an acyclic directed graph with $|V(G)| \geq k$. We prove that there exists a colouring $\{C_1, C_2, \ldots, C_m\}$ such that for every collection $\{P_1, P_2, \ldots, P_k\}$ of k vertex disjoint paths with $|\bigcup_{j=1}^{k} P_j|$ a maximum, each colour class C_i meets $\min\{|C_i|, k\}$ of these paths. An analogous theorem, partially interchanging the roles of paths and colour classes, has been shown by Cameron [4] and Saks [17] and we indicate a third proof.

1. Introduction. Let $G = (V, E)$ be a directed graph containing no loops or multiple edges. A *path* P in G is a sequence of distinct vertices $(v_1, v_2, \ldots, v_l)$ such that $(v_i, v_{i+1}) \in E$, $i = 1, 2, \ldots, l-1$. The set of vertices $\{v_1, v_2, \ldots, v_l\}$ of a path $P = (v_1, v_2, \ldots, v_l)$ will be denoted by $V(P)$. The *cardinality* of P, denoted by $|P|$, is $|V(P)|$.

A family $\mathscr{P}$ of paths is called a *path-partition* of G if its members are vertex disjoint and $\bigcup\{V(P)\colon P \in \mathscr{P}\} = V$. For each nonnegative integer k, the *k-norm* $|\mathscr{P}|_k$ of a path partition $\mathscr{P} = \{P_1, \ldots, P_m\}$ is defined by

$$|\mathscr{P}|_k = \sum_{i=1}^{m} \min\{|P_i|, k\}.$$

A partition which minimizes $|\mathscr{P}|_k$ is called *k-optimum*. For example, a 1-optimum partition is a partition P containing a minimum number of paths.

A *partial k-colouring* is a family $\mathscr{C}^k = \{C_1, C_2, \ldots, C_t\}$ of at most k disjoint independent sets C_i called *colour classes*. The cardinality of a partial k-colouring $\mathscr{C}^k = \{C_1, C_2, \ldots, C_t\}$ is $|\bigcup_{i=1}^{t} C_i|$, and $\mathscr{C}^k$ is said to be *optimum* if $|\bigcup_{i=1}^{t} C_i|$ is as large as possible. A path partition $\mathscr{P} = \{P_1, P_2, \ldots, P_m\}$ and a partial k-colouring $\mathscr{C}^k$ are *orthogonal* if every path P_i in $\mathscr{P}$ meets $\min\{|P_i|, k\}$ different colour classes of $\mathscr{C}^k$.

Berge [2] made the following conjecture:

Conjecture 1. Let G be a directed graph and let k be a positive integer. Then for every k-optimum path partition $\mathscr{P}$, there exists a partial k-colouring $\mathscr{C}^k$ orthogonal to $\mathscr{P}$.