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RETICULATED SETS AND THE ISOMORPHISM OF ANALYTIC POWERS

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We study the properties of separable measurable spaces which are "Borel-dense of order n." Those Borel-dense of order 1 are precisely those that embed as a subset of the unit interval with totally imperfect complement, and the nth order version is an appropriate casting of this idea into n dimensions. The concept enables one to sharpen some known results concerning the isomorphism types of analytic spaces. A result of Mauldin and Shortt (separately) may be stated thus:

(1) If X is a space Borel-dense of order 1 and is Borel-isomorphic with $X \times X$, then X is automatically a standard (absolute Borel) space. (Mauldin assumed X to be analytic.)

We obtain the following enlargement:

(2) If X is a space Borel-dense of order n and X^n is Borel-isomorphic with X^m (some $m > n$), then X is an analytic space.

The requirement of nth order density is not overly severe. Complements (in a standard space) of universally null sets are Borel-dense of every finite order, for example; the same may be said for complements of sets always of first category or, more generally, of sets with Marczewski's property (s^0). Statement 2 might therefore be regarded as a criterion whereby to judge which universally null sets (or sets always of first category, or sets with property (s^0)) are co-analytic. It should also be mentioned, however, that the problem of finding a particular Borel-dense non-Borel analytic space A for which $A^2 \cong A^3$ is open; it may be that "analytic" in statement 2 can be strengthened to "standard". The relationship between Borel-density and the Blackwell property is also noted.

Our method of proof revolves around a strengthening of a classical theorem of Mazurkiewicz and Sierpiński [**10**] to the effect that if A is an analytic subset of a product $S_1 \times S_2$, then the set of s in S_1 such that the section $A(s)$ is uncountable is analytic. A multi-dimensional version of this theorem is proposition 1 *infra*, wherein "uncountable" is replaced with "non-recticulate" in keeping with the dimensions of the sections. The fact that the projection of an analytic set is again analytic is expanded into this multi-dimensional setting in Proposition 2. Other classical results of Lusin and Braun do not generalize, however, as is shown by an example.