PACIFIC JOURNAL OF MATHEMATICS
Vol. 119, No. 1, 1985

DEGENERATE SECANT VARIETIES AND A PROBLEM ON MATRICES

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We show that if a developable ruled surface of a curve in complex projective space has a degenerate secant variety, then the surface already lies in a $\mathbb{P}^4$. This result eliminates a redundancy in the list of Griffiths and Harris, of surfaces that have degenerate secant varieties.

1. Introduction. A d-dimensional variety $X \subset \mathbb{P}^N_{\mathbb{C}}$, $N \geq 2d+1$, is said to have a degenerate secant variety, $\mathrm{Sec}(X)$, when $\dim(\mathrm{Sec}(X)) \leq 2d$. In [**1**, Results 5.37, 6.16–18], Griffiths and Harris prove the Proposition: Let $X \subset \mathbb{P}^N$ be a surface having a degenerate secant variety. Then either (i) $X \subset \mathbb{P}^4$, (ii) X is a cone, (iii) X is the Veronese surface, or (iv) X is developable.

It is easy to show that any of conditions (i), (ii) or (iii) implies that X has a degenerate secant variety. The main contribution of the present paper is the Proposition (3.5): If a developable surface, X, has a degenerate secant variety, then X is contained in a $\mathbb{P}^4$.

Combining the above two results, we obtain

THEOREM (1.0). *A surface $X \subset \mathbb{P}^N$ has a degenerate secant variety precisely when one of conditions* (i), (ii) *or* (iii) *above is satisfied.*

It is interesting to note that a developable surface always has a degenerate tangent variety i.e.

$$\dim(\mathrm{Tan}(X)) \leq 3 \qquad [\mathbf{1}, \text{Result } 5.37];$$

this also follows from our Lemma (3.3).

We will see in §3 that a variety $X^d \subset \mathbb{P}^N$ gives rise to a family of $(N-d) \times (d+1)$ matrices $\{A(x)\}$ where x belongs to some polydisc $U \subset \mathbb{C}^d$. Let $|\ |$ denote "determinant". Result (3.2.1) states that

$$\begin{aligned} &\dim \mathrm{Sec}(X) \leq 2d \\ &\Leftrightarrow \mathrm{rank}(A(x) - A(y)) \leq d \quad \forall x, y \in U \\ &\Leftrightarrow |A^I(x) - A^I(y)| = 0 \quad \text{for all } (d+1)\text{-tuples of rows } I = (i_0, i_1, \ldots, i_d). \end{aligned}$$