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CODIMENSION TWO ISOMETRIC IMMERSIONS BETWEEN EUCLIDEAN SPACES

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Hartman and Nirenberg showed that any C^∞ isometric immersion f: $\mathbf{E}^n \to \mathbf{E}^{n+1}$ between flat Euclidean spaces is a cylinder erected over a plane curve. We show that in the codimension two case, f: $\mathbf{E}^n \to \mathbf{E}^{n+2}$ factors as a composition of isometric immersions $f = f_1 \circ f_2$: $\mathbf{E}^n \to \mathbf{E}^{n+1} \to \mathbf{E}^{n+2}$, when $n > 1$ and f has nowhere zero normal curvature. Counterexamples are given if this assumption is relaxed.

How can paper be folded? More precisely, how can flat Euclidean 2-space $\mathbf{E}^2$ be isometrically immersed into flat Euclidean n-space $\mathbf{E}^n$ (for simplicity, assume C^∞ differentiability everywhere). For $n = 3$, A. V. Pogorelov [**4**] announced without proof that the image is a cylinder erected over a plane curve; proofs may be found in Massey [**3**] and Stoker [**5**]. In this paper, we consider $n = 4$ and show that any isometric immersion g: $\mathbf{E}^2 \to \mathbf{E}^4$ with nowhere zero normal curvature factors as a composition of isometric immersions $g = g_1 \circ g_2$: $\mathbf{E}^2 \to \mathbf{E}^2 \to \mathbf{E}^4$.

The result of Pogorelov has been generalized by Hartman and Nirenberg [**2**]. They showed that the image of any codimension-one isometric immersion between flat Euclidean spaces is a cylinder erected over a plane curve. Using a result of Hartman [**1**] we easily show that any codimension-two, isometric immersion f: $\mathbf{E}^n \to \mathbf{E}^{n+2}$, $n > 1$, with nowhere zero normal curvature factors as a composition $f = f_1 \circ f_2$: $\mathbf{E}^n \to \mathbf{E}^{n+1} \to \mathbf{E}^{n+2}$. The images of f_1 and f_2 are cylinders. The assumption of nowhere zero normal curvature is essential; counterexamples are given in §3 when the assumption is relaxed.

From another point of view, the cylinders of Pogorelov and Hartman and Nirenberg can be deformed ("unrolled") through a one-parameter family of isometric immersions to a hyperplane. This family is obtained by deforming the generating plane curve to a straight line. From our results, it follows easily that any isometric immersion f: $\mathbf{E}^n \to \mathbf{E}^{n+2}$ with nowhere zero normal curvature can be deformed through isometric immersions to a standard inclusion i: $\mathbf{E}^n \hookrightarrow \mathbf{E}^{n+2}$ (it would be interesting to know if the normal curvature assumption can be removed). In addition, we proved [**7**] that if the normal curvature is identically zero, then any