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NON-COMPACT SETS WITH CONVEX SECTIONS

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Two further generalizations of Ky Fan's generalizations of his well-known intersection theorem concerning sets with convex sections are obtained.

1. Introduction. Let I be an index set; in the case when I is finite, it is always assumed that I contains at least two indices. Let $\{X_i\}_{i\in I}$ be a family of topological spaces and $X := \prod_{i\in I} X_i$. For each $i \in I$, set

$$X^i := \prod_{\substack{j\neq i\\ j\in I}} X_j \qquad \left(\text{so that } X = X_i \times X^i\right),$$

and let $p_i\colon X \to X_i$ and $p^i\colon X \to X^i$ be the projections. For each $x \in X$, we write $p_i(x) = x_i$ and $p^i(x) = x^i$. For any non-empty subset K of X, we let $p_i(K) = K_i$ and $p^i(K) = K^i$.

Our aim in this paper is to give two generalizations of the following intersection theorem of Ky Fan [2] concerning sets with convex sections.

THEOREM 1. (*Ky Fan.*) *Let $X_1, X_2, \ldots, X_n$ be n (≥ 2) non-empty compact convex sets each in a Hausdorff topological vector space. Let $X := \prod_{i=1}^{n} X_i$ and $A_1, A_2, \ldots, A_n$ be n subsets of X such that*

(a) *For each $i = 1, 2, \ldots, n$ and any $x_i \in X_i$, the section*

$$A_i(x_i) := \left\{x^i \in X^i\colon \left(x_i, x^i\right) \in A_i\right\}$$

is open in X^i.

(b) *For each $i = 1, 2, \ldots, n$ and any $x^i \in X^i$, the section*

$$A_i(x^i) := \left\{x_i \in X_i\colon \left(x_i, x^i\right) \in A_i\right\}$$

is convex and non-empty.

Then the intersection $\bigcap_{i=1}^{n} A_i$ is non-empty.

Theorem 1 is a unified account of game-theoretic results for arbitrary n-person games and has several applications [2], [3]. In particular, Tychonoff's fixed point theorem [11], Sion's generalization [10] of von Neumann's minimax principle [8] and Nash's equilibrium point theorem [7] are immediate consequences of Theorem 1.