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THE DIOPHANTINE EQUATION $ax + by = c$ IN $Q(\sqrt{5})$ AND OTHER NUMBER FIELDS

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Solving in rational integers the linear diophantine equation

(1) $\qquad ax + by = c, \qquad (a, b)|c, a, b, c, \in Z$

is very well known. Let $d = (a, b)$, **and put** $A = a/d, B = b/d, C = c/d$, **then equation (1) becomes**

(1′) $\qquad Ax + By + C, \qquad (A, B) = 1, A, B, C, \in Z.$

The purpose of this note is to discuss the solutions of this equation when A, B, C **are integers in** $Q(\sqrt{5})$ **and the solutions are integers in** $Q(\sqrt{5})$. **What makes the discussion interesting is that an algorithm which mimics the continued fraction algorithm that solves the rational integer case can be implemented.**

A brief summary of the continued fraction algorithm for the rational case is as follows: To solve (1′): find the regular simple continued fraction for A/B; i.e.

$$\frac{A}{B} = r_0 + \cfrac{1}{r_1 + \ddots + \cfrac{1}{r_n}}$$

which we write as $A/B = (r_0; r_1, \ldots, r_n)$. Since A/B is rational, the continued fraction is finite. The $(m + 1)$th convergent of a continued fraction is denoted by $P_m/Q_m = (r_0; r_1 \cdots r_m)$. If $A/B = P_n/Q_n$ then the penultimate convergent P_{n-1}/Q_{n-1} provides a solution to $Ax + By = 1$ because of the well-known relation.

(2) $$P_nQ_{n-1} - Q_nP_{n-1} = (-1)^{n+1}.$$

It suffices therefore to take $x = (-1)^{n+1}Q_{n-1}, y = (-1)^nP_{n-1}$. To solve (1) we take $x = (-1)^{n+1}dCQ_{n-1}$ and $y = (-1)^{n+1}dCP_{n-1}$.

It is well known that the integers in $Q(\sqrt{5})$ have the form $s + t\lambda$, where $s, t \in Z$ and $\lambda = (1 + \sqrt{5})/2$. (See Hardy and Wright [1] or Niven and Zuckerman [3] for a complete discussion of this algebraic number field.) The elements in $Q(\sqrt{5})$ are of course the quotients of integers in the