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RANDOM PERMUTATIONS AND BROWNIAN MOTION

J. M. DELAURENTIS AND B. G. PITTEL

Consider the cycles of the random permutation of length n. Let $X_n(t)$ be the number of cycles with length not exceeding n^t, $t \in [0,1]$. The random process $Y_n(t) = (X_n(t) - t\ln n)/\ln^{1/2} n$ is shown to converge weakly to the standard Brownian motion $W(t)$, $t \in [0,1]$. It follows that, as a process, the empirical distribution function of "loglengths" of the cycles weakly converges to the Brownian Bridge process. As another application, an alternative proof is given for the Erdös-Turán Theorem: it states that the group-order of random permutation is asymptotically $e^{\mathscr{U}}$, where $\mathscr{U}$ is Gaussian with mean $\ln^2 n/2$ and variance $\ln^3 n/3$.

1. Introduction. Results. Consider S_n, the symmetric group of permutations of a set $\{1,\dots,n\}$ endowed with the uniform distribution, $P(\sigma) = 1/n!$ for each $\sigma \in S_n$. Since a pioneering work by Goncharov [**10**], [**11**], a considerable attention has been paid to the asymptotic study of the order sequence of cycles lengths for the random permutation (r.p.), and of components sizes for the random mapping (Kolchin, et al. [**13**], [**14**], Shepp and Lloyd [**20**], Balakrishnan, et al. [**1**], Stephanov [**21**], Vershik and Shmidt [**22**]). Let $X_{ns} = X_{ns}(\sigma)$ designate the random number of cycles of length s in the r.p. σ. It is known [**11**] that X_n, the total number of cycles, it asymptotically normal with mean and variance $\ln n$. A similar result holds true for the total number of cycles whose lengths are divisible by a given number, [**4**], [**20**]. In this paper, we study the asymptotical behavior of the *joint* distribution of $X_{n1},\dots,X_{nn}$.

For each $t \in [0,1]$, consider

$$(1.1)\quad X_n(t) = \sum_{1\le s\le n^t} X_{ns}, \qquad Y_n(t) = \big(X_n(t) - t\ln n\big)/\ln^{1/2} n;$$

so, $X_n(t)$ is the total number of cycles of the r.p. with lengths not exceeding n^t. Clearly, each sample function of $Y_n(\cdot)$ belongs to $D[0,1]$ the space of functions on $[0,1]$ which are right-continuous at each $t \in [0,1)$ and have left limits at each $t \in (0,1]$. Introduce $W(t)$, $t \in [0,1]$, the standard Brownian motion defined on a complete probability space with continuous sample paths. Let $\mathscr{H}$ be a class of functionals on $D[0,1]$ continuous in the sup-norm metric.