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CARDINALITY CONSTRAINTS FOR PSEUDOCOMPACT AND FOR TOTALLY DENSE SUBGROUPS OF COMPACT TOPOLOGICAL GROUPS

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Let K be a compact, Hausdorff topological group, $\mathscr{G}(K)$ the set of dense, pseudocompact subgroups of K, and $m(K) = \min\{|G|: G \in \mathscr{G}(K)\}$. We show: (1) $m(K)$ is a function of the weight of K (in the sense that if K' is another such group with $w(K) = w(K')$, then $m(K) = m(K')$); and (2) if K is connected then every totally dense subgroup D of K satisfies $|D| = |K|$. With these results in hand we classify (a) those cardinals α such that $m(K) < |K|$ when $w(K) = \alpha$ and (b) those cardinals α such that some compact K with $w(K) = \alpha$ admits a totally dense subgroup D with $|D| < |K|$. The conditions of (a) and (b) are incompatible in some models of ZFC (e.g., under GCH) and are compatible in others. Thus the following question, the origin of this work, is undecidable in ZFC: Is there a compact, Hausdorff, topological group K with a totally dense, pseudocompact subgroup G such that $|G| < |K|$?

1. Notation and conventions. We denote the least infinite cardinal number by the symbol ω.

Let α be an infinite cardinal. We denote by α^+ the least cardinal β such that $\beta > \alpha$, and we denote by $\mathrm{cf}(\alpha)$ the least cardinal γ such that there exists a family $\{\alpha_i: i \in I\}$ with $|I| = \gamma$ for which each $\alpha_i < \alpha$ and $\sum_{i\in I} \alpha_i = \alpha$.

The symbols **Z**, **Q**, **R** and **T** denote respectively the sets of integers, of rational numbers, of real numbers, and of complex numbers of modulus 1. In each case we assume when convenient the usual algebraic and topological properties.

By a space, or a topological space, we mean a completely regular, Hausdorff space, i.e., a Tychonoff space. By a topological group we mean an ordered triple $G = \langle G, \circ, \mathscr{T}\rangle$ such that

(i) $\langle G, \circ\rangle$ is a group,

(ii) $\langle G, \mathscr{T}\rangle$ is a topological space, and

(iii) the function $\langle x, y\rangle \to xy^{-1}$ is continuous from $G \times G$ to G.

It is well known (see for example **[28]** (8.4)) that if $\langle G, \circ, \mathscr{T}\rangle$ satisfies (i) and (iii) and the T_0 separation property, then $\langle G, \mathscr{T}\rangle$ is a Tychonoff space.