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CONTINUITY OF HOMOMORPHISMS OF BANACH G-MODULES

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We consider whether, given a locally compact abelian group G and two Banach G-modules X and Y, every G-module homomorphism from X into Y is continuous. Discontinuous homomorphisms can exist only when Y has submodules on which G acts by scalar multiplication. They are also associated with discontinuous convariant forms on X so if either of these are absent them all G-module homomorphisms are continuous.

1. Introduction. Throughout this paper G is a locally compact abelian group.

DEFINITION 1.1. A *Banach G-module* is a Banach space X with a map $(g, x) \mapsto gx$ of $G \times X$ into X such that

(i) $x \mapsto gx$ is linear on $X\,(g \in G)$.

(ii) $g(hx) = (gh)x \quad (g, h \in G, x \in X)$.

(iii) $ex = x \quad (x \in X$, e is the identity element of G).

(iv) There is a $K \in \mathbf{R}$ with

$$\|gx\| \leq K\|x\| \qquad (x \in X, g \in G).$$

Note that we do not require any continuity of the map $(g, x) \mapsto gx$ in g—in fact in most of the paper we will be treating G as a discrete group.

A *G-submodule* of X is a closed linear subspace X_0 of X with $gx \in X_0$ $(g \in G, x \in X_0)$. The G-module X is *scalar* if for each $g \in G$ there is $\lambda(g) \in \mathbf{C}$ with $gx = \lambda(g)x$ $(g \in G, x \in X)$. If $X \neq \{0\}$ then $\lambda(e) = 1$, $\lambda(gh) = \lambda(g)\lambda(h)$ and $|\lambda(g)| \leq K$. Applying this last inequality to g^n $(n \in \mathbf{Z})$ we see $|\lambda(g)| = 1$ so λ is a character and mild continuity hypotheses on $g \mapsto gx$ would imply that λ is continuous.

DEFINITION 1.2. Let X, Y be Banach G-modules. Then S: $X \to Y$ is a *G-module homomorphism* if it is linear and $S(gx) = gS(x)$ $(g \in G, x \in X)$.

If Y is a scalar module then $S(gx) = \lambda(g)\,S(x)$ and we say that S is λ-covariant. In the special case when $\lambda \equiv 1$ is the trivial character we say S is invariant. When $Y = \mathbf{C}$ we call S a *form*.

Invariant and covariant forms are related in many cases because if S is a λ covariant form on X and T: $X \to X$ is a linear map with