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WEIGHTS AND $L \log L$

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In memory of Irving L. Glicksberg
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Let $\omega(x)$ be a positive locally integrable weight on $[0,1]$. Discussed are conditions on ω necessary and sufficient for the (dyadic) Hardy-Littlewood maximal function to map $L \log L(w\,dx)$ into $L^1(\omega\,dx)$ or into weak L^1.

1. Introduction. Let Mf denote the (dyadic) Hardy-Littlewood maximal function of f, for f locally integrable on $\mathbf{R}^n$. That is,

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f|,$$

the sup being taken over all dyadic cubes in $\mathbf{R}^n$ containing x. It is well-known that M is a bounded operator from $L^p(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ when $p > 1$, takes L^1 to weak L^1, and for functions f supported in a dyadic cube Q_0 satisfies

$$\int_{Q_0} Mf \le C \int_{Q_0} |f| \log^+ |f| + C|Q_0|.$$

More recently, Muckenhoupt and others have studied the behaviour of M when the L^p spaces with respect to Lebesgue measure are replaced by those with respect to the measure $\omega(x)\,dx$, $\omega \in L^1_{\text{loc}}$. A nonnegative locally integrable function ω is said to be in Muckenhoupt's (dyadic) A_p class for $1 \le p < \infty$ if

$$\sup_{Q \text{ dyadic}} \left\| \frac{\omega_Q}{\omega(x)} \right\|_{L^q(\omega\,dx/\omega(Q))} = A_p(\omega) = A_p < \infty. \tag{1.1}$$

Here $1/p + 1/q = 1$, ω_Q is the average $(1/|Q|)\int_Q \omega$ of ω over Q and $\omega(Q) = \int_Q \omega$. (More generally, if E is a measurable set in $\mathbf{R}^n$ we denote $\int_E \omega$ by $\omega(E)$.) In [3], Muckenhoupt proved that given $\omega \in L^1_{\text{loc}}$ there exists a constant $C = C_{p,\omega,n}$ such that

$$\int |Mf|^p \omega \le C \int |f|^p \omega \tag{1.2}$$

if and only if ω is in the A_p class, and that $\omega\{Mf > \lambda\} \le (C/\lambda)\int |f|\omega$ if and only if ω is in the A_1 class. It would therefore seem reasonable to