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ON THE BOUNDARY CONTINUITY OF CONFORMAL MAPS

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Let the function f map the unit disk $\mathbf{D}$ conformally onto the domain G in $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. The prime end theory of Carathéodory gives a completely geometric characterization of the boundary behavior of f. Prime ends are defined in terms of crosscuts of G.

Our aim is to give a geometric description of the boundary behavior of f that refers only to the boundary ∂G and not to the domain itself. It can therefore be applied to any complementary domain of a connected closed set in $\hat{\mathbf{C}}$. Our description will however be incomplete because we will have to allow exceptional sets.

1. Introduction and results. We say that f has the *angular limit* $f(\zeta)$ at $\zeta \in \partial\mathbf{D}$ if

$$f(\zeta) = \lim_{z \to \zeta,\, z \in \Delta} f(z) \in \hat{\mathbf{C}}$$

exists for every Stolz angle Δ at ζ; we shall always denote by $f(\zeta)$ the angular limit if it exists. A theorem of Beurling [**1**] (see e.g. [**4**, p. 56] [**8**, p. 341, 344]) states that the angular limit $f(\zeta)$ exists for $\zeta \in B$ where $\operatorname{cap}(\partial\mathbf{D} \setminus B) = 0$ and furthermore that

$$\operatorname{cap}\{\zeta \in B : f(\zeta) = \omega\} = 0 \quad \text{for } \omega \in \hat{\mathbf{C}};$$

here cap denotes the logarithmic capacity.

We shall say that f is *continuous at* $\zeta \in \partial\mathbf{D}$ if f has a continuous extension to $\mathbf{D} \cup \{\zeta\}$, that is, if $f(z) \to f(\zeta)$ as $z \to \zeta$, $z \in \mathbf{D}$. Our first result states that discontinuity tends to imply injectivity.

THEOREM 1. *Let f map $\mathbf{D}$ conformally onto G. Then there is a partition*

$$\partial\mathbf{D} = A_0 \cup A_1 \cup A_2 \tag{1.1}$$

such that

(i) $\operatorname{cap} A_0 = 0$,

(ii) *the angular limit $f(\zeta)$ exists for every $\zeta \in A_1$, and f is one-to-one on A_1,*

(iii) *f is continuous at each $\zeta \in A_2$, and f is exactly two-to-one on A_2.*