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PAIRS OF POSITIVE SOLUTIONS OF QUASILINEAR ELLIPTIC EQUATIONS IN EXTERIOR DOMAINS

TAKAŜI KUSANO, CHARLES A. SWANSON AND HIROYUKI USAMI

Our main objective is to prove the existence of infinitely many pairs (u_1, u_2) of positive solutions of quasilinear elliptic differential equations

$$\Delta u - q(|x|)u = f(x, u, \nabla u), \qquad x \in \Omega_\alpha, \tag{1.1}$$

throughout exterior domains $\Omega_\alpha \subset \mathbf{R}^N$, $N \geq 2$, of the type

$$\Omega_\alpha = \{x \in \mathbf{R}^N : |x| > \alpha\}, \qquad \alpha > 0,$$

where $x = (x_1, \ldots, x_N)$, $\nabla u = (\partial u/\partial x_1, \ldots, \partial u/\partial x_N)$, and $\Delta = \nabla \cdot \nabla$. Each pair has the property that $u_1(x)/u_2(x)$ has uniform limit zero in Ω_α as $|x| \to \infty$. In particular, if $q(t) \equiv 0$ and $N \geq 3$, $u_1(x)$ has limit 0 as $|x| \to \infty$, and $u_2(x)$ is bounded above and below by positive constants in Ω_α.

1. The function q in (1.1) is assumed to be nonnegative and locally Hölder continuous in $\mathbf{R}_+ = [0, \infty)$, and $f\colon \Omega_\alpha \times \mathbf{R}_+ \times \mathbf{R}^N \to \mathbf{R}$ is locally Hölder continuous in $\Omega_\alpha \times \mathbf{R}_+ \times \mathbf{R}^N$ and satisfies a Nagumo condition. Detailed hypotheses are listed in §3.

Specific asymptotic estimates for the growth (decay) of the solutions $u_1(x)$, $u_2(x)$ as $|x| \to \infty$ follow easily from our construction. In particular sufficient conditions are given for the quasilinear equation

$$\Delta u - |x|^{2r} u = \phi(x) u^\gamma + \psi(x) |\nabla u|^\beta, \qquad x \in \Omega_\alpha,$$

for constants $r \geq 0$, $\gamma \geq 0$, $0 \leq \beta \leq 2$, to have positive solutions $u_1(x)$, $u_2(x)$ in Ω_α such that $u_i(x)$ is bounded above and below by positive constant multiples of

$$|x|^{-\lambda} \exp\left[(-1)^i |x|^{r+1}/(r+1)\right], \qquad x \in \Omega_\alpha,\ i = 1, 2,$$

where $\lambda = (N + r - 1)/2$.

By a similar method we also prove the existence of infinitely many positive solutions of the boundary value problem

$$\begin{aligned} &\Delta u - q(|x|)u = f(x, u, \nabla u), \qquad x \in \Omega_\alpha, \\ &u|_{\partial\Omega_\alpha} = 0 \end{aligned} \tag{1.2}$$

under the same hypotheses as for (1.1).