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GROTHENDIECK LOCALLY CONVEX SPACES OF CONTINUOUS VECTOR VALUED FUNCTIONS

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Let $\mathscr{C}(X, E)$ be the space of continuous functions from the completely regular Hausdorff space X into the Hausdorff locally convex space E, endowed with the compact-open topology. Our aim is to characterize the $\mathscr{C}(X, E)$ spaces which have the following property: weak-star and weak sequential convergences coincide in the equicontinuous subsets of $\mathscr{C}(X, E)'$. These spaces are here called Grothendieck spaces. It is shown that in the equicontinuous subsets of E' the $\sigma(E', E)$- and $\beta(E', E)$-sequential convergences coincide, if $\mathscr{C}(X, E)$ is a Grothendieck space and X contains an infinite compact subset. Conversely, if X is a G-space and E is a strict inductive limit of Fréchet-Montel spaces $\mathscr{C}(X, E)$ is a Grothendieck space. Therefore, it is proved that if E is a separable Fréchet space, then E is a Montel space if and only if there is an infinite compact Hausdorff X such that $\mathscr{C}(X, E)$ is a Grothendieck space.

1. Introduction. In this paper X will always denote a completely regular Hausdorff topological space, E a Hausdorff locally convex space, and $\mathscr{C}(X, E)$ the space of continuous functions from X into E, endowed with the compact-open topology. When E is the scalar field of reals or complex numbers, we write $\mathscr{C}(X)$ instead $\mathscr{C}(X, E)$.

It is well known that $\mathscr{C}(X, E)$ is a Montel space whenever $\mathscr{C}(X)$ and E so are, hence, if and only if X is discrete and E is a Montel space (see [**5**], [**16**]).

We study what happens when X has the following weaker property: the compact subsets of X are G-spaces (see below for definitions).

We obtain in Theorem 4.4 that if E is a Fréchet-Montel space and X has that property, then $\mathscr{C}(X, E)$ is a Grothendieck locally convex space. The key in the proof is the following fact: every countable equicontinuous subset of $\mathscr{C}(X, E)'$ lies, via a Radon-Nikodým theorem, in a suitable $L^1(\tau, E'_\beta)$. As a consequence of a theorem of Mújica [**10**], the same result is true when E is a strict inductive limit of Fréchet-Montel spaces.

In §3 we study the converse of 4.4. In Corollary 3.3 it is proved that if X contains an infinite compact subset, E is a Fréchet separable space and $\mathscr{C}(X, E)$ is a Grothendieck space, then E is a Montel space. This property characterizes the Montel spaces among the Fréchet separable spaces.