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SEMIPRIME ℵ-QF 3 RINGS

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A ring R (associative with identity) is called *right* ℵ-QF 3 if it has a faithful right ideal which is a direct sum of a family of injective envelopes of pairwise non-isomorphic simple right R-modules. A right QF 3 ring is just a right ℵ-QF 3 ring where the above family is finite. The aim of the present work is to give a structure theorem for semiprime ℵ-QF 3 rings. It is proved, among others, that the following conditions are equivalent for a given ring R: (a) R is a semiprime right ℵ-QF 3 ring, (b) there is a ring Q, which is a direct product of right full linear rings, such that $\operatorname{Soc} Q \subset R \subset Q$, (c) R is right nonsingular and every non-singular right R-module is cogenerated by simple and projective modules.

A ring R is called a *right* QF 3 ring if there is a minimal faithful module U_R, in the sense that every faithful right R-module contains a direct summand which is isomorphic to U; one proves that if there exists such a module U, then it is unique up to an isomorphism. It was proved by Colby and Rutter [**5**, Theorem 1] that R is right QF 3 if and only if it contains a faithful right ideal of the form $E(S_1) \oplus \cdots \oplus E(S_n)$, where each $E(S_i)$ is the injective envelope of a simple module S_i, and the S_i's are pairwise non-isomorphic. Following Kawada [**10**], we say that R is a *right* ℵ-QF 3 ring if there is a family $(e_\lambda)_{\lambda\in\Lambda}$ of pairwise orthogonal and pairwise non isomorphic (in the sense that $e_\lambda R \neq e_\mu R$ whenever $\lambda \neq \mu$) idempotents of R such that: (a) each $e_\lambda R$ is the injective envelope of a minimal right ideal, (b) the right ideal $W_R = \sum_{\lambda\in\Lambda} e_\lambda R$ is faithful; here ℵ stands for the cardinality of the set Λ. It is clear from Colby and Rutter's result that a right QF 3 ring is nothing other than a right ℵ-QF 3 ring where ℵ is a finite cardinal. By a ℵ-QF 3 ring we shall mean a ring which is both right and left ℵ-QF 3; similarly for QF 3 rings.

In [**4**] we studied those right ℵ-QF 3 rings which have zero right singular ideal. Our purpose in the present paper is to characterize the semiprime right ℵ-QF 3 rings. Our main result is that the following conditions are equivalent for a given ring R: (a) R is a semiprime right ℵ-QF 3 ring, (b) R is a semiprime ring with essential socle and every simple projective right R-module is injective. (c) R is right nonsingular and every nonsingular right R-module is cogenerated by simple projective modules, (d) R is (isomorphic to) a subring of a direct product $\prod_{\lambda\in\Lambda} Q_\lambda$ of right full linear rings and $\bigoplus_{\lambda\in\Lambda} \operatorname{Soc} Q_\lambda \subset R$. As a consequence we obtain that R is a semiprime ℵ-QF 3 ring if and only if it satisfies one (and hence all) of the following conditions: (a) R is a subring of the direct