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REGULAR OPERATOR APPROXIMATION THEORY

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Regular operator approximation theory applies to finite difference approximations for differential equations and numerical integration approximations for integral equations. New relationships and efficient derivations of known results are presented. The analysis is based on the systematic use of convergence and compactness properties of sequences of sets. Since the purpose is theoretical, applications are merely indicated and references are cited.

1. Introduction and Summary. Regular operator approximation theory applies to numerical solutions of differential and integral equations. Some pertinent references are Anselone and Ansorge [**3, 4**], Chatelin [**6**], Grigorieff [**7, 8**], Stummel [**10**], and Vainikko [**12**].

We focus here on linear equations in a Banach space setting. Approximations are defined in the same setting. This case serves to motivate ideas and results for the more complicated situation with approximations defined in different spaces related by connection maps, such as restriction and interpolation. We present particularly efficient and revealing derivations of basic convergence results and we extend the theory in several significant respects. The analysis is based on systematic use of convergence and compactness concepts for sequences of sets. Some of these ideas were exploited in nonlinear operator approximation theory in [**3**]. These concepts should be useful also in general approximation theory.

Let X and Y be Banach spaces. A sequence of elements in X or Y is discretely compact (d-compact for short) if every subsequence has a convergent subsequence.

Let $A, A_n \in L(X, Y)$ the space of bounded linear operators from X to Y. We shall compare equations

$$Ax = y, \qquad A_n x_n = y,$$

where $A_n \to A$ pointwise and $\{A_n\}$ is asymptotically regular, i.e., if $\{x_n\}$ is bounded and $\{A_n x_n\}$ is d-compact, then $\{x_n\}$ is d-compact. Results concern inverse operators, null spaces, and ranges. There are implications for eigenvalues and eigenvectors.

Sharper results can be given for equations

$$(I - K)x = y, \qquad (I - K_n)x_n = y,$$