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COUNTEREXAMPLE TO A CONJECTURE OF H. HOPF

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The purpose of this paper is to produce an immersion of a compact oriented two-dimensional surface of genus one into Euclidean 3-space with constant mean curvature $H \neq 0$. We thus provide a counterexample in dimension 3 to the following conjecture of H. Hopf.

Conjecture of H. Hopf. Let Σ be an immersion of an oriented, closed hypersurface with constant mean curvature $H \neq 0$ in R^n. Must Σ be the standard embedded $(n-1)$-sphere?

Two important results relating to this conjecture are due to A. D. Alexandrov and H. Hopf. A. D. Alexandrov [**1**] showed that the conjecture is true if Σ is an embedded hypersurface in R^n. This extended an old result of J. H. Jellett [**10**] (see also [**15**] p. 354), who showed the conjecture to be valid in the case where Σ is a two-dimensional star-shaped surface in R^3. H. Hopf himself [**8**] showed the conjecture to be true when Σ is an immersion of S^2 into R^3 with constant mean curvature.

A negative answer to the Hopf conjecture in dimensions greater than three was recently supplied by Wu-Yi Hsiang [**9**]. He constructed a counterexample in R^4. He considered 3-dimensional immersions into R^4 which were invariant under the action of $O(2) \times O(2)$, a subgroup of the isometry group for R^4. If one identifies R^4 with $C \times C$ so that a point in R^4 has coordinates (z_1, z_2) where $z_i = x_i + iy_i$ and the action of $O(2) \times O(2)$ to be given by $(z_1, z_2) \to (e^{i\theta}z_1, e^{i\alpha}z_2)$, then the orbit space is $R^4/O(2) \times O(2) = \{(x_1, x_2) | x_1 \geq 0, x_2 \geq 0\}$ and a surface of constant mean curvature with the desired symmetry is determined by a generating curve lying in the orbit space. Such a curve will generate a closed surface if it terminates on the positive x_1 and x_2 axes. Hsiang succeeded in showing that there exist such curves which generate an immersion of S^3 into R^4 of constant mean curvature which is not a standard sphere. This method does not carry over to the classical dimension and so the Hopf conjecture for R^3 remains unresolved.

Our counterexample is contained in the following theorem.

COUNTEREXAMPLE THEOREM. *There is a conformal immersion of R^2 into R^3 with constant mean curvature $H \neq 0$ which is doubly-periodic with respect to a rectangle in R^2. If $w = u + iv = (u, v)$ represents a typical*