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ON WEAK EPIMORPHISMS IN HOMOTOPY THEORY

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Weakened versions of the categorical notions of epimorphism and monomorphism have proved to be of some interest in pointed homotopy theory. A weak epimorphism, for instance, is a morphism e (in any category with 0 objects) such that $g \circ e = 0$ implies $g = 0$.

In 1967, Ganea utilized extensive homotopy-theoretic calculations to exhibit examples, in the pointed homotopy category, of weak monomorphisms which are not monomorphisms. In this note, we exploit the properties of a remarkable group discovered by Higman in 1951 to exhibit examples, again in the pointed homotopy category, of weak epimorphisms which are not epimorphisms, thereby confirming a suspicion enunciated by Hilton in the early 1960's.

1. A *weak epimorphism* in a category with 0 objects is a morphism $X \xrightarrow{e} Y$ satisfying *weak right cancellation*: if $Y \xrightarrow{g} Z$ is a morphism such that $g \circ e = 0$, then $g = 0$. The notion of weak epimorphism, as well as the dual notion of weak monomorphism (a morphism satisfying weak left cancellation), arises rather naturally in the pointed homotopy category $\mathscr{H}$ of topological spaces (compare [**R**]; in [**R**], the objects in $\mathscr{H}$ are taken to be path-connected CW-spaces, but this restriction is unncessary here) and our purpose here is to compare this notion with the more traditional notion of homotopy-epimorphism (= epimorphism in $\mathscr{H}$).

A study of the comparison between weak monomorphisms and monomorphisms in homotopy theory was carried out by Ganea [**G**] who, in particular, answered a number of questions raised in Hilton's notes [**H2**]. An additional problem hinted at in [**H2**; p. 180] is settled here; we show that weak epimorphisms in $\mathscr{H}$ are, in general, genuinely weaker than epimorphisms in $\mathscr{H}$.

Before proceeding with the details, a few remarks linking the notions of (weak) epimorphism and (weak) monomorphism in $\mathscr{H}$ may be in order. By definition, a (weak) monomorphism $X \to Y$ induces a (weak) monomorphism of pointed morphism sets $[W, X] \to [W, Y]$ for each W in $\mathscr{H}$. Similarly, a (weak) epimorphism $X \to Y$ induces a (weak) monomorphism of pointed morphism sets $[Y, Z] \to [X, Z]$ for each Z in $\mathscr{H}$. Sharper statements are provided by the following.