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A NORMAL FORM AND INTEGRATION IN FINITE TERMS FOR A CLASS OF ELEMENTARY FUNCTIONS

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Let C be the field of complex numbers, E the usual exponential on C. So (C, E) is an exponential field.

We define an exponential ring extension $\mathbf{C}\{x\}^E$ of (C, E) and give a functional representation: $\mathbf{C}\{x\}^E$ is isomorphic to the smallest exponential ring extension of (C, E) containing the functions x^s, x a real and positive variable, and $s \in \mathbf{C}$.

Finally, we give a simple integration-in-finite-terms algorithm for elements of $\mathbf{C}\{x\}^E$.

Introduction. An "exponential ring" (R, E) or "E-ring" is an algebraic structure with R a commutative ring with 1 and E: $R \to R$ (the "exponential") a map such that

$$E(a+b) = E(a)E(b) \quad \text{for all } a, b \in R$$
$$E(0) = 1.$$

A "derivation" on (R, E) is a map D: $R \to R$ such that for all $a, b \in R$ we have:

$$D(a+b) = Da + Db$$
$$Dab = (Da)b + a(Db)$$
$$DE(a) = E(a)Da.$$

In [5] the concept of "ring of E-polynomials" in the indeterminate x is defined and denoted "$R[x]^E$". It is an exponential ring extension of R, generated as an exponential ring by the set $R \cup \{x\}$. A unique derivation D can be defined on $R[x]^E$ such that $Dr = 0$ for $r \in R$, and $Dx = 1$. (More generally, the ring $R[x_1, \ldots, x_n]^E$ is defined in a similar way in [5] but we will not deal with it here.)

In §1, we define an exponential differential ring extension $R\{x\}^E$ of a given (R, E). The ring $R[x]^E$ is the exponential subring of $R\{x\}^E$ generated by the set $R \cup \{x\}$. It should be remarked however that $R[x]^E$ contains an isomorphic copy of $R\{x\}^E$, namely $R[E(x)]^E$ which is the exponential subring generated by $R \cup \{E(x)\}$.