PACIFIC JOURNAL OF MATHEMATICS
Vol. 121, No. 2, 1986

ON SINGULARITY OF HARMONIC MEASURE IN SPACE

JANG-MEI WU

We construct a topological ball D in $\mathbf{R}^3$, and a set E on ∂D lying on a 2-dimensional hyperplane so that E has Hausdorff dimension one and has positive harmonic measure with respect to D. This shows that a theorem of Øksendal on harmonic measure in $\mathbf{R}^2$ is not true in $\mathbf{R}^3$. Suppose D is a bounded domain in $\mathbf{R}^m$, $m \geq 2$, $\mathbf{R}^m \setminus D$ satisfies the corkscrew condition at each point on ∂D; and E is a set on ∂D lying also on a BMO_1 surface, which is more general than a hyperplane; then we can prove that if E has $m-1$ dimensional Hausdorff measure zero then it must have harmonic measure zero with respect to D.

Lavrentiev (1936) found a simply-connected domain D in $\mathbf{R}^2$ and a set E on ∂D which has zero linear measure and positive harmonic measure with respect to D [5]. McMillan and Piranian subsequently simplified the example [6]. See also [1] and [3].

In [7], Øksendal proved that if D is a simply-connected domain in $\mathbf{R}^2$, and E is a set on ∂D with vanishing linear measure, and if E is situated on a line, then E has vanishing harmonic measure $\omega(E, D)$ with respect to D. In [3], Kaufman and Wu generalized this result and proved that the theorem still holds if E is situated on a quasi-smooth curve, but no longer holds if E is situated on a quasi-conformal circle. An interesting, perhaps very difficult, question is whether the theorem is true if E lies on a rectifiable curve.

Another question is the higher dimensional generalization: if D is a topological ball in $\mathbf{R}^m$, $m \geq 3$, and E is a set on ∂D, situated also on an $m-1$ dimensional hyperplane, does the vanishing of the $m-1$ dimensional Hausdorff measure, $\Lambda^{m-1}(E) = 0$, imply that $\omega(E, D) = 0$?

We answer this negatively by giving the following example.

EXAMPLE. There exists a topological ball D in $\mathbf{R}^3$, and a set E on ∂D, lying on a 2-dimensional hyperplane so that E has Hausdorff dimension one but has positive harmonic measure with respect to D.

We notice that $\dim E = 1$ is much stronger than $\Lambda^2(E) = 0$; and that 1 is best possible, because if $\dim E < 1$ then E has zero capacity in $\mathbf{R}^3$, hence E has zero harmonic measure with respect to D in $\mathbf{R}^3$.