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ON VALUED, COMPLETE FIELDS AND THEIR AUTOMORPHISMS

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The following theorem is proved:

THEOREM. *Let K be a valued, complete field, and assume that the valuation topology admits countable neighbourhood bases. All the automorphisms of K are continuous if and only if K is not algebraically closed.*

Introduction. In this paper we study the following problem: Which valued, complete fields have the property that all their automorphisms are continuous? In the case of Archimedean valuations the answer is well known: There are only two complete fields, $\mathbf{R}$ and $\mathbf{C}$. $\operatorname{Aut}(\mathbf{R}) = \{\mathrm{Id}\}$; $\mathbf{C}$ has exactly two continuous automorphisms, but card $\operatorname{Aut}(\mathbf{C}) = 2^{(2^{\aleph_0})}$, hence very many $\tau \in \operatorname{Aut}(\mathbf{C})$ are discontinuous. Henceforth we shall consider exclusively fields K with non-Archimedean valuations ϕ in the sense of Krull. We assume that the valuation topology admits countable neighbourhood bases. Suppose now that (K, ϕ) is complete. Our main result is that all the automorphisms of K are continuous if and only if K is not algebraically closed. Thus there is a perfect analogy to the Archimedean case.

If the valuation ϕ has rank 1, Hensel's lemma is available and a rather short proof can be given (see §4). The case where ϕ has infinite rank requires a new method; our proof in §5 will be based on a lemma on solvability of certain infinite systems of equations. The technique of this lemma can be applied to other problems on complete fields; two illustrations for this are given in §6.

We should like to mention that the paper was motivated by studies in the theory of orthomodular spaces. These are, by definition, vector spaces E endowed with a hermitian form Ψ such that the lattice $L = \{U \subseteq E: U = (U^{\perp})^{\perp}\}$ of all orthogonally closed subspaces of (E, Ψ) satisfies the orthomodular law: $U \leq V \Rightarrow V = U \vee (V \wedge U^{\perp})$ for all $U, V \in L$. Classical examples are the Hilbert spaces over $\mathbf{R}$ or $\mathbf{C}$. In recent years numerous non-classical, infinite dimensional orthomodular spaces have been discovered. All these new spaces are constructed over certain valued,