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THE COLLARS OF A RIEMANNIAN MANIFOLD AND STABLE ISOSYSTOLIC INEQUALITIES

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We define the p-dimensional collar $\mathrm{Col}_p(M, g)$ of a compact torsion-free Riemannian manifold (M, g) to be the greatest lower bound of the masses of all the p-dimensional currents which represent non-trivial integral homology classes. When the cohomology ring of M satisfies a certain non-degeneracy condition there is an inequality giving a lower bound on the volume of (M, g) in terms of certain p-dimensional collars of (M, g). This is a version of the stable isosystolic inequality using currents rather than singular homology.

In addition to deriving this version of the stable isosystolic inequality, we show for one class of manifolds that it is a sharp inequality.

THEOREM A. *Let (M, g) be a compact oriented n-dimensional Riemannian manifold with $H_1(M, Z) \cong Z$. Then*

$$\mathrm{Vol}(M, g) \geq \mathrm{Col}_1(M, g)\mathrm{Col}_{n-1}(M, g).$$

Furthermore, equality holds if and only if there is a Riemannian submersion of (M, g) onto the circle of length $\mathrm{Col}_1(M, g)$ such that each level hypersurface (i.e. fiber) is a connected minimal submanifold of volume $\mathrm{Col}_{n-1}(M, g)$.

It is interesting to contrast Theorem A with Loewner's inequality [**2**], [**8**] which gives a lower bound on the area of a torus in terms of the length of the shortest non-contractible closed curve. In Loewner's theorem equality holds for a class of metrics which differ from one another by a constant multiple. Whereas in Theorem A equality can hold for many very different Riemannian metrics. As an example let $M = S^1 \times S^2$. Certainly the equation $\mathrm{Vol}(M, g) = \mathrm{Col}_1(M, g)\mathrm{Col}_2(M, g)$ will hold for any of the various product metrics g. It also will hold for some non-product metrics. One of the latter can be constructed as follows. Let S^2 be given the canonical constant curvature metric, and let f be a non-trivial orientation preserving isometry of S^2. Then the group of integers acts as a properly discontinuous group of isometries on the Riemannian product $R \times S^2$ by defining $n(t, x) = (t + n, f^n(x))$ where f^n is the nth iterate of f. The quotient space under this action is diffeomorphic to M. Hence the metric on $R \times S^2$ passes down to a non-product metric on M for which (1) the