PACIFIC JOURNAL OF MATHEMATICS
Vol. 121, No. 2, 1986

ON PROJECTIONS OF REAL ALGEBRAIC VARIETIES

C. ANDRADAS AND J. M. GAMBOA

In this paper we generalize an earlier result of the authors, showing that any closed semialgebraic set whose Zariski-closure is irreducible, is the projection under a finite map of an irreducible real algebraic set (see Theorem 3.2 below).

1. Introduction. This result is, somehow, striking and shows the wild nature of irreducible real algebraic sets; no matter how complicated (in terms of connected components, pieces of different dimensions, etc...) a closed semialgebraic set is (as far as its Zariski-closure is irreducible, which is a fairly weak condition), there exists an irreducible algebraic set projecting onto it. In this way, some examples of "exotic" algebraic sets are obtained in §4.

Also in §4, as an application, is stated a result on the Harrison's topology of the space of orders of function fields, which in fact was the starting point of this work. The problem is: Is any clopen (i.e. closed and open) subset of the space of orders of a function field the image of the space of orders of a finite extension? This question is proposed in **[E-L-W]**. The answer is affirmative (see 4.1 below) but as it is usual dealing with orders in function fields we had to translate the problem to geometry using the, well known at this moment, correspondence between clopens and semialgebraic sets of a model of the function field. In particular some results of **[D-R]** are needed. Thus the question follows as a corollary of the geometric result 3.2.

Throughout the paper R denotes an arbitrary real closed field. Given any real algebraic set V we denote by V_c the set of central points of V, that is the closure of the regular points of V. We work always with the order topology of R^n. Finally the symbol π always stands for the projection into the n first coordinates.

2. The key proposition. Here V will be an irreducible algebraic subset of R^n. We shall denote by $\mathfrak{p}$ its ideal polynomials and by $R[V]$ its coordinate ring.