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REPRESENTATIONS OF TERNARY QUADRATIC FORMS AND THE CLASS NUMBER OF IMAGINARY QUADRATIC FIELDS

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In this paper, we consider the norm form of a definite rational quaternion algebra restricted to the elements of trace zero in a maximal order of the algebra. When the algebra has class number one, we derive an equation which relates the representation numbers of the norm form to the class number of imaginary quadratic extensions of the rational numbers.

0. Introduction. Kneser [**8**] observed that the existence of a relation between these two quantities is not unexpected. When compared to the Dirichlet class number formula, the Minkowski-Siegel formulas suggest a connection between the weighted average of the number of primitive representations of an integer m by the different forms in the genus of a given definite ternary quadratic form and the number of ideal classes in an order of $\mathbf{Q}(\sqrt{-m})$ (e.g. see [3] Appendix B). This connection is evidenced by comparing the local p-factors in each formula. In the case that the genus consists of only one class, one derives information about the representation numbers of the given form. However, this approach has two disadvantages. First, the job of determining the p-factors for primes p dividing twice the discriminant of the form is at best awkward, and second, such an analytic proof would not provide as explicit a correspondence between ideal classes and primitive elements as the one given by the arithmetic approach which we shall use.

In [**6**], Gauss showed that the number of primitive integral solutions (i.e. $x, y, z \in \mathbf{Z}$ and $(x, y, z) = 1$) to $x^2 + y^2 + z^2 = m$ is a constant multiple of the class number of primitive binary quadratic forms of discriminant $-4m$; the constant is 12 or 24 depending only on the congruence class of m modulo 4. In the 1920's, Venkov [**15**] elegantly reproved Gauss' result by viewing the ternary form as the (reduced) norm of a generic element of trace zero in the maximal order

$$\Lambda = \mathbf{Z}\left(\frac{1+i+j+k}{2}\right) + \mathbf{Z}i + \mathbf{Z}j + \mathbf{Z}k$$

(Hurwitz's quaternions) in the quaternion algebra $(\frac{-1,-1}{\mathbf{Q}})$. Rehm [**11**] recently reproved some of Venkov's results in a more modern framework.