PACIFIC JOURNAL OF MATHEMATICS
Vol. 122, No. 1, 1986

ABELIAN GROUPS AND PACKING BY SEMICROSSES

DEAN HICKERSON AND SHERMAN STEIN

Motivated by a question about geometric packings in n-dimensional Euclidean space, $\mathbf{R}^n$, we consider the following problem about finite abelian groups. Let n be an integer, $n \geq 3$, and let k be a positive integer. Let $g(k, n)$ be the order of the smallest abelian group in which there exist n elements, $a_1, a_2, \ldots, a_n$, such that the kn elements ia_j, $1 \leq i \leq k$, are distinct and not 0. We will show that for n fixed, $g(k, n) \sim 2\cos(\pi/n)k^{3/2}$.

The geometric question concerns certain star bodies, called "semicrosses", which are defined as follows:

If k and n are positive integers, a (k, n)-semicross consists of $kn + 1$ unit cubes in $\mathbf{R}^n$, a "corner" cube parallel to the coordinate axes together with n arms of length k attached to faces of the cube, one such arm pointing in the direction of each positive coordinate axis. Let K, the "semicross at the origin", be the semicross whose corner cube is $[0,1]^n$. Then every semicross is a translate of K; i.e. has the form $v + K$ for some vector v.

A family of translates $\{v + K\colon v \in H\}$ is called an integer lattice packing if H is an n-dimensional subgroup of Z^n and, for any two vectors v and w in H, the interiors of $v + K$ and $w + K$ are disjoint. We shall examine how densely such packings pack $\mathbf{R}^n$ for large k, and show that, for $n \geq 3$, this density is asymptotic to

$$\frac{n \sec \pi/n}{2\sqrt{k}}.$$

(For $n = 1$ or 2 the density is 1 for every k.)

This result contrasts with the already known result for crosses. (A (k, n)-cross consists of $2kn + 1$ unit cubes, a center cube with an arm of length k attached to each face.) As shown in **[St1]**, for $n \geq 2$ the integer lattice packing density of the (k, n)-cross is asymptotic to $2n/k$.

0. Preliminary matters. Suppose M is a set of nonzero integers, G is an abelian group, and n is a positive integer. We say that M n-packs G if there is a set $S \subseteq G$ such that $|S| = n$ and the elements ms with $m \in M$ and $s \in S$ are distinct and nonzero. Such a set S is called a packing set.